Part I

Long-range-order (LRO) and Spontaneous Symmetry Breaking (SSB) in classical and quantum systems

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LRO and SSB appear universally in a wide range of systems with a large degree of freedom.

spin systems are most suitable for understanding the essence of the phenomena

- equlibirum $\begin{cases} \text{classiacal} \\ \text{quantum} \end{cases}$ similar; classiacal → $\boxed{\text{Ising model}}$

- ground state $\begin{cases} \text{cl\cancel{asstal}} \quad \text{trivial} \\ \text{quantum} \rightarrow \boxed{\text{Heisenberg model}} \end{cases}$

⟨Phase transition, LRO and SSB in the Ising model⟩

§ Definitions.

set of sites   set of bonds $\boxed{\text{d-dim. hyper cubic lattice}}$

lattice $(\Lambda_L, \mathcal{B}_L)$ L even.

$$\begin{cases} \Lambda_L := \{(x_1, \dots, x_d) \mid x_i \in \mathbb{Z}, -\frac{L}{2} < x_i \leq \frac{L}{2}\} \subset \mathbb{Z}^d \\ \mathcal{B}_L := \{(x, y) \mid x, y \in \Lambda_L, \ |x-y| = 1\} \end{cases}$$

use periodic b.c

$(x, y) = (y, x)$

spin variable $\sigma_x \in \{-1, 1\}$, $x \in \Lambda_L$

$$\boldsymbol{\sigma} = (\sigma_x)_{x \in \Lambda_L} \in \{-1, 1\}^{\Lambda_L}$$

Hamiltonian $H(\boldsymbol{\sigma}) = - \sum_{(x,y) \in \mathcal{B}_L} \sigma_x \sigma_y$

omit

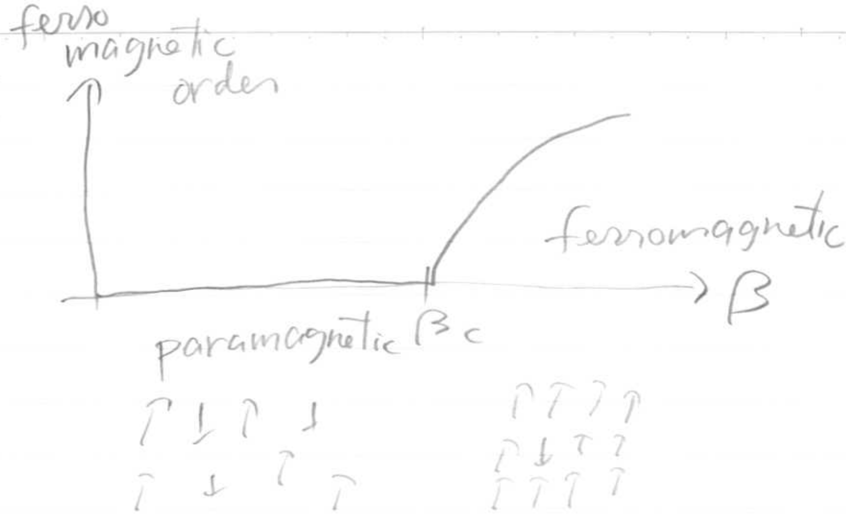
equilibrium at $\beta > 0$

any function of $\boldsymbol{\sigma}$

$$\langle F \rangle_{\beta, L} := \frac{1}{Z_L(\beta)} \sum_{\boldsymbol{\sigma}} F e^{-\beta H}$$

$$Z_L(\beta) = \sum_{\boldsymbol{\sigma}} e^{-\beta H}$$

We expect $d \geq 2$



relevant symmetry global spinflip $\sigma \to -\sigma$

order parameter $\mathcal{O} = \sum_{x \in \Lambda_L}' \sigma_x$ (total magnetization)

$$\langle \mathcal{O} \rangle_{\beta, L} = 0 \text{ by the symmetry for } \forall \beta, L$$

external field $h \geq 0$

$$H_h = H - h\mathcal{O}$$

$$\langle F \rangle_{\beta, h, L} := \frac{\sum_{\sigma}' F e^{-\beta H_h}}{\sum_{\sigma}' e^{-\beta H_h}}$$

§ High temperatures → disordered.

we want a proof that works for $\forall L$ ⇒ easy

there are many methods.

<u>Lemma</u> For any $x \neq y \in \Lambda_L$

$$0 \leq \langle \sigma_x \sigma_y \rangle_{\beta, L} \leq \beta \sum_{\substack{z \in \Lambda_L \\ (|z-x|=1)}}' \langle \sigma_z \sigma_y \rangle_{\beta, L} \qquad (h=0)$$

upperbound for the "propergation" of interaction

of course $\langle \sigma_x \sigma_y \rangle = 1$ $\quad x = y$

So

$$\langle \sigma_x \sigma_y \rangle \leq \beta \sum_{\substack{z \\ (|z-x|=1)}}' \langle \sigma_z \sigma_y \rangle \leq \beta^2 \sum_{\substack{z, z' \\ (|z-x|=|z'-z|=1)}}' \langle \sigma_{z'} \sigma_y \rangle$$

$$\cdots \leq \sum_{\omega : x \to y}' \beta^{|\omega|}$$

ω : random walk from x to y

$$\omega = (z_0, z_1, \ldots, z_n)$$

$$z_0 = x, \quad z_1 = y, \quad |z_i - z_{i+1}| = 1, \quad |\omega| = n$$

$$\therefore \langle \sigma_x \sigma_y \rangle \leq \sum_{n=0}^{\infty}{}' \beta^n N_{x \to y}(n)$$

└ the number of $\omega_{x \to y}$ s.t. $|\omega| = n$

Clearly $N_{x \to y}(n) \begin{cases} = 0 & n \leq |x-y| \\ \leq (2d)^n & n \geq |x-y| \end{cases}$

So
$$\langle \sigma_x \sigma_y \rangle \leq \sum_{n=|x-y|}^{\infty} (2d\beta)^n = \frac{(2d\beta)^{|x-y|}}{1-2d\beta}$$

$\boxed{\text{if } 2d\beta < 1}$ $(\beta < (2d)^{-1})$

$$= \text{const } e^{-\frac{|x-y|}{\xi(\beta)}}$$

← exp. decay in $|x-y|$!

$((2d)^{-1} < \beta c)$

note also that

$$\sum_{y \in \Lambda_L} \langle \sigma_x \sigma_y \rangle \leq \sum_{\omega : x \to} \beta^{|\omega|} \leq \sum_{n=0}^{\infty} (2d\beta)^n = \frac{1}{1-2d\beta}$$

any walk from x

$\therefore$

$$\left\langle \left(\frac{\Theta}{L^d} \right)^2 \right\rangle_{\beta,L} = \frac{1}{L^{2d}} \sum_{x,y \in \Lambda_L} \langle \sigma_x \sigma_y \rangle \leq \frac{1}{L^d (1-2d\beta)}$$

$$\boxed{\lim_{L \uparrow \infty} \left\langle \left(\frac{\Theta}{L^d} \right)^2 \right\rangle_{\beta,L} = 0 \quad \text{if } \beta < (2d)^{-1}}$$

consistent

We also have

$$\boxed{\left\langle \left(\frac{\Theta}{L^d} \right) \right\rangle_{\beta,L} = 0 \quad \text{for } {}^\forall L, {}^\forall \beta}$$

proof of the Lemma

$\mathcal{N} := \{ w \mid w \in \Lambda_L ,\ |x-w| = 1 \}$ $\qquad H = -\sum_{w\in\mathcal{N}} \sigma_x \sigma_w + H'$

$$Z\langle \sigma_x \sigma_y \rangle = \sum_{\sigma} \sigma_x \sigma_y e^{-\beta H}$$

$$= \sum_{\sigma} \sigma_x \sigma_y \Big(\prod_{w\in\mathcal{N}} \underbrace{e^{\beta \sigma_x \sigma_w}}_{= \sum_{n_w=0}^{\infty} \frac{(\beta \sigma_x \sigma_w)^{n_w}}{n_w!}} \Big) e^{-\beta H'}$$

$$= \sum_{\mathbf{n}} \frac{\beta^{\sum n_w}}{\prod_w n_w!} \underbrace{\sum_{\sigma} \sigma_x^{1+\sum_w n_w} \Big(\prod_w (\sigma_w)^{n_w} \Big) \sigma_y e^{-\beta H'}}$$

all n_w: $\mathbf{n} = (n_w)_{w\in\mathcal{N}}$

$$= \underbrace{\sum_{\sigma_x = \pm 1} \sigma_x^{1+\sum n_w}}_{0 \text{ or } 2} \underbrace{\sum_{\sigma'} \Big\{ \Big(\prod \sigma_w^{n_w} \Big) \sigma_y e^{-\beta H'} \Big\}}_{\geq 0}$$

$$\sum_{\sigma_x = \pm 1} \sigma_x^{1+\sum n_w} \leq \left(\sum n_w \right) \sum_{\sigma_x = \pm 1} \sigma_x^{(\sum n_w) - 1}$$

equality if Gaussian

$$\leq \sum_{z\in\mathcal{N}} \sum_{\mathbf{n}} n_z \frac{\beta^{\sum n_w}}{\prod_w n_w!} \sum_{\sigma} \sigma_x^{(\sum n_w)-1} \Big(\prod_w (\sigma_w)^{n_w} \Big) \sigma_y e^{-\beta H'}$$

def.

$\tilde{\mathbf{n}} = (\tilde{n}_w)_{w\in\mathcal{N}}$ by $\tilde{n}_w = \begin{cases} n_w - 1 & w = z \\ n_w & w \neq z \end{cases}$

$$= \beta \sum_{z \in \mathcal{N}}{}' \sum_{\tilde{m}}{}' \frac{\beta^{\sum \tilde{n}_\omega}}{\prod_\omega \tilde{n}_\omega !} \sum_{\sigma}{}' \left(\prod_\omega (\sigma_x \sigma_\omega)^{\tilde{n}_\omega} \right) \underbrace{\sigma_z}_{\substack{\varphi \\ \text{extra}}} \sigma_y \, e^{-\beta H'}$$

$$= \beta \sum_{z \in \mathcal{N}}{}' \sum_{\sigma}{}' \sigma_z \sigma_y \, e^{-\beta H} = \beta \sum_{z \in \mathcal{N}}{}' Z \langle \sigma_z \sigma_y \rangle$$

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Use similar method to prove

$$\langle \sigma_x \rangle_{\beta, h, L} \leq \frac{\beta h}{1 - 2d\beta}$$

for any $h \geq 0$ and $\beta \leq (2d)^{-1}$.

§ Low temperatures → spins are ordered.
we need a cleverer argument

Theorem (a variant of the theorems by Peierls 1936, Griffiths 1964, Dobrushin 1965)

independent of L.

$d \geq 2$ $\exists \beta_0(d)$, $\exists q_0(\beta) > 0$ for $\beta > \beta_0(d)$

we have $\langle \sigma_x \sigma_y \rangle_{\beta,L} \geq q_0(\beta)$

for $\forall L$, $\forall x, y \in \Lambda_L$, and $\forall \beta > \beta_0(d)$

LRO spins align with each other

$\therefore$

$$\langle \mathcal{O}^2 \rangle_{\beta,L} = \sum_{x,y\in\Lambda_L} \langle \sigma_x \sigma_y \rangle_{\beta,L} \geq q_0(\beta) L^{2d}$$

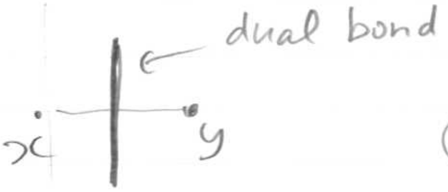
standard way of expressing LRO.

stochastic geometric representation

Proof of the theorem (only for $d=2$)

unhappy bond

given σ, draw the dual bond of $(x,y)\in\mathcal{B}$ if $\sigma_x\sigma_y=-1$



$$\sigma \longrightarrow G$$

a closed graph of dual bonds → boundary between + and −

2 to 1

$\pm\sigma$ gives the same G

$$H(\sigma) = -\{-|G| + (\underbrace{|\mathcal{B}|}_{=2L^2}-|G|)\} = 2|G| - 2L^2$$

penalty for having the boundary

$$\therefore\ Z = 2e^{2\beta L^2}\sum_G e^{-2\beta|G|}$$

x, y sep. by an even number of "walls" in G

$$Z\langle\sigma_x\sigma_y\rangle = 2e^{2\beta L^2}\Big\{\sum_G I[x \overset{\text{even}}{\sim} y]\, e^{-2\beta|G|} - \sum_G I[x \overset{\text{odd}}{\sim} y]\, e^{-2\beta|G|}\Big\}$$

$I[\text{true}]=1$
$I[\text{false}]=0$

$$= 2e^{2\beta L^2}\sum_G \{1-2I[x \overset{\text{odd}}{\sim} y]\}\, e^{-2\beta|G|}$$

$\therefore$

$$\langle\sigma_x\sigma_y\rangle = 1 - 2\,\frac{\sum_G I[x \overset{\text{odd}}{\sim} y]\, e^{-2\beta|G|}}{\sum_G e^{-2\beta|G|}}$$

$$I[x \overset{\text{odd}}{\sim} y] \leq I[\exists \text{ a loop } \gamma \subset G \text{ which separates } x \text{ and } y]$$

Crude

$$\leq \sum_{\gamma}{}' I[\gamma \subset G]$$

(γ: simple loop, γ separates x&y)

we also count γ

We need to show that γ is rare when β is large.

$$\therefore \langle \sigma_x \sigma_y \rangle \geq 1 - 2 \sum_{\gamma}{}' P(\gamma)$$

$(\ldots)$

where

$$P(\gamma) = \frac{\sum_G' I(\gamma \subset G) e^{-2\beta|G|}}{\sum_G' e^{-2\beta|G|}}$$

$\tilde{G} = G \setminus \gamma$ is also a closed graph

$$\sum_G{}' I[\gamma \subset G] e^{-2\beta|G|} = e^{-2\beta|\gamma|} \sum_{\substack{\tilde{G} \\ (\tilde{G} \cap \gamma = \emptyset)}}{}' e^{-2\beta|\tilde{G}|}$$

$$\sum_G{}' e^{-2\beta|G|} \geq \sum_{\substack{\tilde{G} \\ (\tilde{G} \cap \gamma = \emptyset)}}{}' e^{-2\beta|\tilde{G}|}$$

$$\therefore \boxed{P(\gamma) \leq e^{-2\beta|\gamma|}}$$ uniform in the size L !

$$\langle \sigma_x \sigma_y \rangle \geq 1 - 2 \sum_{\gamma}{}' e^{-2\beta|\gamma|} = 1 - 2 \sum_{n=4}^{\infty}{}' N(n) e^{-2\beta n}$$

(γ: closed loop sep. x&y)

$$N(n) \leq 2n^2 3^n$$

$$\geq 1 - 2 \sum_{n=4}^{\infty}{}' n^2 (3e^{-2\beta})^n$$

for suff. large β. $= q(\beta) > 0$ ~~for suff~~

<u>Rem.</u> Similar proof for $d \geq 3$ is possible but not straightforward.

But $\langle \sigma_x \sigma_y \rangle_{d\text{-dim}} \geq \langle \sigma_x \sigma_y \rangle_{2\text{-dim}}$ $(d \geq 3)$

But still from the symmetry

$$\langle \sigma_x \rangle_{\beta, L} = 0 \quad \text{for } {}^\forall L.$$

$$\therefore \quad \langle \mathcal{O} \rangle_{\beta, L} = 0$$

No magnetic order ?

$$\boxed{\text{LRO without SSB}}$$

§From LRO to SSB

<u>Theorem</u> (Griffiths 1966)

Suppose that $\langle \mathcal{O}^2 \rangle_{\beta,L} \geq q_0 L^{2d}$ with $q_0 > 0$ for $\forall L$

Then $\lim_{h\downarrow 0} \lim_{L\uparrow\infty} \frac{1}{L^d} \langle \mathcal{O} \rangle_{\beta,h,L} \geq \sqrt{q_0}$

(to be rigorous, replace lim with liminf)

SSB the symmetry $(\mathcal{O} \to -\mathcal{O})$ is broken even when $h\downarrow 0$

infinitessimally small h broke the sym

<u>Corollary</u> $d \geq 2$, $\beta \geq \beta_0(d)$, $\lim_{h\downarrow 0} \lim_{L\uparrow\infty} \frac{1}{L^d} \langle \mathcal{O} \rangle_{\beta,h,L} \geq \sqrt{q_0(\beta)}$

$$\boxed{\text{LRO (with } h=0) \Longrightarrow \text{SSB (as } h\downarrow 0)}$$

+ ~~LRO~~ $\to \langle \mathcal{O}^2 \rangle \geq \langle \mathcal{O} \rangle^2$ in any situation

<u>Rem.</u> long-range order parameter $q(\beta) = \lim_{L\uparrow\infty} \frac{1}{L^{2d}} \langle \mathcal{O}^2 \rangle_{\beta,L}$

spontaneous magnetization (order parameter) $m^*(\beta) = \lim_{h\downarrow 0} \lim_{L\uparrow\infty} \frac{1}{L^d} \langle \mathcal{O} \rangle_{\beta,h,L}$

Griffiths $m^*(\beta) \geq \sqrt{q(\beta)}$ (quite general)

for the Ising model

$m^*(\beta) = \sqrt{q(\beta)}$ has been proved basic idea

prob. dist. of the magnetization for $h=0$

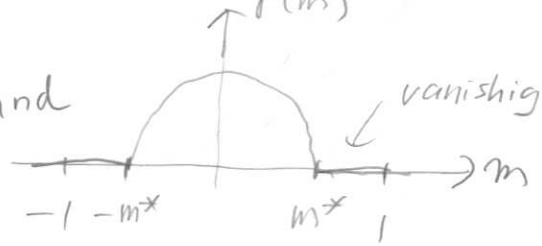
$$P_L(m) = \frac{1}{Z_L(\beta)} \sum_{\sigma} I\left[\frac{\Theta}{L^d}=m\right] e^{-\beta H}$$

$(-1 \leq m \leq 1)$

then

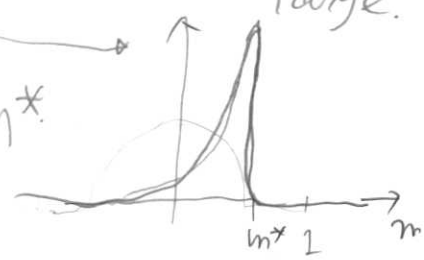
$$\frac{1}{L^d}\langle\Theta\rangle_{\beta,h,L} = \frac{\sum_m m\, P_L(m)\, e^{\beta h L^d m}}{\sum_m P_L(m)\, e^{\beta h L^d m}}$$

suppose $P_L(m) \xrightarrow[L\to\infty]{} P(m)$ and



$h>0$

$P_L(m)\, e^{\beta h L^d m}$ has a peak very close to m^* if L is large.



So $\displaystyle\lim_{h\downarrow 0}\lim_{L\to\infty} \frac{1}{L^d}\langle\Theta\rangle_{\beta,h,L} = m^*$

It is also obvious that

$$\left\langle\left(\frac{\Theta}{L^d}\right)^2\right\rangle_{\beta,L} = \sum_m m^2 P_L(m) \leq (m^*)^2$$

$$\therefore\ q(\beta) \leq (m^*)^2$$

if we have $P(m) \simeq \frac{1}{2}\{\delta(m-m^*) + \delta(m+m^*)\}$

then $q(\beta) = (m^*)^2$ ($\leftarrow$ Ising)

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Proof of the Theorem

$p_L(m)$

Suppose $\sum_m{}' m^2 p(m) > q$

take $0 < \varepsilon < \sqrt{q}/2$

$$q \le \sum_m{}' m^2 p(m) = \sum_{\substack{m \\ (|m|<\sqrt{q}-\varepsilon)}}{}' m^2 p(m) + 2 \sum_{\substack{m \\ (m \ge \sqrt{q}-\varepsilon)}}{}' m^2 p(m)$$

$$\le (\sqrt{q}-\varepsilon)^2 + 2 \sum_{\substack{m \\ (m \ge \sqrt{q}-\varepsilon)}}{}' p(m)$$

$$\therefore \sum_{\substack{m \\ (m \ge \sqrt{q}-\varepsilon)}}{}' p(m) \ge \frac{1}{2}\{q - (\sqrt{q}-\varepsilon)^2\} =: \alpha(\varepsilon) > 0$$

$=: A = A_1 + A_2$

$V = L^d$

$$\langle m_L \rangle_{\beta,h,L} = \frac{\sum_m{}' m\, p(m)\, e^{\beta h V m}}{\sum_m{}' p(m)\, e^{\beta h V m}}$$

$=: B = B_1 + B_2$

$$B_1 := \sum_{\substack{m \\ (m \ge \sqrt{q}-2\varepsilon)}}{}' p(m)\, e^{\beta h V m} \ge \sum_{\substack{m \\ (m \ge \sqrt{q}-\varepsilon)}} p(m)\, e^{\beta h V m} \ge \alpha(\varepsilon)\, e^{\beta h V(\sqrt{q}-\varepsilon)}$$

$$B_2 := \sum_{\substack{m \\ (m \le \sqrt{q}-2\varepsilon)}}{}' p(m)\, e^{\beta h V m} \le e^{\beta h V(\sqrt{q}-2\varepsilon)} \le \frac{e^{-\beta h V \varepsilon}}{\alpha(\varepsilon)} B_1$$

$$\therefore B \le \left\{1 + \frac{e^{-\beta h V \varepsilon}}{\alpha(\varepsilon)}\right\} B_1$$

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$$A_1 := \sum_{\substack{m \\ (m \geq \sqrt{q}-2\varepsilon)}}' m\, P(m) e^{\beta h V m} \geq (\sqrt{q}-\varepsilon)\,\alpha(\varepsilon)\, e^{\beta h V(\sqrt{q}-\varepsilon)}$$

$$A_2 := \sum_{\substack{m \\ (m < \sqrt{q}-2\varepsilon)}}' m\, P(m) e^{\beta h V m} \geq -1 \geq -\frac{e^{-\beta h V(\sqrt{q}-\varepsilon)}}{(\sqrt{q}-\varepsilon)\,\alpha(\varepsilon)} A_1$$

$$\therefore \quad A \leq \left\{ 1 - \frac{e^{-\beta h V(\sqrt{q}-\varepsilon)}}{(\sqrt{q}-\varepsilon)\,\alpha(\varepsilon)} \right\} A_1$$

So $h>0$

$$\langle m_L \rangle_{\beta,h,L} \geq \frac{1 - \dfrac{e^{-\beta h V(\sqrt{q}-\varepsilon)}}{(\sqrt{q}-\varepsilon)\,\alpha(\varepsilon)}}{1 + \dfrac{e^{-\beta h V \varepsilon}}{\alpha(\varepsilon)}} \; \underbrace{\frac{\sum'_{\substack{m \\ (m \geq \sqrt{q}-2\varepsilon)}} m\, P(m) e^{\beta h V m}}{\sum'_{\substack{m \\ (m \geq \sqrt{q}-2\varepsilon)}} P(m) e^{\beta h V m}}}_{\geq \sqrt{q}-2\varepsilon}$$

$\therefore$

$$\liminf_{L \uparrow \infty} \langle m_L \rangle_{\beta,h,L} \geq \sqrt{q} - 2\varepsilon$$

for $\forall h > 0$

$\varepsilon \downarrow 0$.

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Rem. The argument works whenever the probability distribution $P_L(m)$ of the order par. is well-defined

↓

The theorem extends automatically to quantum systems where $[\hat{H}, \hat{\mathcal{O}}] = 0$

$\hat{\mathcal{O}}$ ← order par.

§ Some remarks about equlibrium states ~~in~~ for the infinite system.

F: arbitrary function (i.e. polynomial) of a finite number of σ_x's $(x \in \mathbb{Z}^d)$

$$\omega_\beta^0(F) := \lim_{L\uparrow\infty} \underbrace{\langle F\rangle_{\beta,L}}_{\text{can be defined for suff. large } L} \quad \leftarrow h=0$$

the existence can be proved (difficult)

$\omega_\beta^0 : \mathfrak{O} \to \mathbb{R}$ an equlibrium state for the Ising model on $\mathbb{Z}^d$.

$\mathfrak{O}$: the set of all F

Rem. H, $Z_L(\beta)$ are NOT defined in $L\uparrow\infty$,
But the state is

For β large $d\geq 2$

for large L.

$$\omega_\beta^0\left(\frac{\mathcal{O}}{L^d}\right) = 0, \quad \omega_\beta^0\left(\left(\frac{\mathcal{O}}{L^d}\right)^2\right) \simeq q > 0$$

Thus the fluct of $\mathcal{O}/L^d$

$$\sqrt{\omega_\beta^0\left(\left(\frac{\mathcal{O}}{L^d}\right)^2\right) - \left(\omega_\beta^0\left(\frac{\mathcal{O}}{L^d}\right)\right)^2} \simeq \sqrt{q} > 0$$

$$\Downarrow$$

inconsistent with thermodynamics!

$\parallel$

the density of a macroscopic quantity should not fluctuate

The state ω_β^0 does not describe a physically realistic equilibrium state.

(math: ω_β^0 is not ergodic) later (2-23)

Define other equilibrium states $\omega_\beta^\pm$ by the existence can be proved

$$\begin{cases} \omega_\beta^+(F) := \lim_{h\downarrow 0}\lim_{L\uparrow\infty} \langle F\rangle_{\beta,h,L} \\ \omega_\beta^-(F) = \lim_{h\uparrow 0}\lim_{L\uparrow\infty} \langle F\rangle_{\beta,h,L} \end{cases}$$

$\beta<\beta_c$ $\omega_\beta^0 = \omega_\beta^+ = \omega_\beta^-$

Then we know ($d \geq 2$, β large)

$$\omega_\beta^\pm\left(\frac{\Theta}{L^d}\right) = \pm m^* \qquad \omega_\beta^\pm\left(\left(\frac{\Theta}{L^d}\right)^2\right) \xrightarrow[L\uparrow\infty]{} q = (m^*)^2$$

$(m^*>0)$

$$\therefore \lim_{L\uparrow\infty}\sqrt{\omega_\beta^\pm\left(\left(\frac{\Theta}{L^d}\right)^2\right) - \left\{\omega_\beta^\pm\left(\frac{\Theta}{L^d}\right)\right\}^2} = 0$$

Θ/L^d has vanishing fluctuation as $L\uparrow\infty$.

The states $\omega_\beta^\pm$ describe realistic equilibrium (pure TD phase)

(math: $\omega_\beta^\pm$ is ergodic)

Theorem (Bodineau 2006), $\forall d$, $\forall \beta > 0$

$$\boxed{\omega_\beta^0(\cdot) = \frac{1}{2}\left\{\omega_\beta^+(\cdot) + \omega_\beta^-(\cdot)\right\} \quad (*)}$$

at low temperatures
unphysical (non-ergodic) state is decomposed

into a mixture of physical (ergodic) states!

↓

general theories
- decomposition into ergodic states
- 〃 pure states

Suppose we did know ~~θ~~ or magnetic field.

the only "natural" eq. state ω_β^0

$\boxed{\text{LRO without SSB.}}$ $\Leftrightarrow$ non ergodic.

⇓

decompostion → $\omega_\beta^\pm$
we get

↑

states with $\boxed{\text{LRO AND SSB}}$

@ ~~We need~~ You need not know h !!

DATE

§ Classical Heisenberg model

spin $\mathbf{S}_x = (S_x^{(1)}, S_x^{(2)}, S_x^{(3)}) \in \mathbb{R}^3$ $\quad \sum_{\alpha=1}^{3} (S_x^{(\alpha)})^2 = 1$

$$H = -\sum_{(x,y)\in\mathcal{B}_L} \mathbf{S}_x \cdot \mathbf{S}_y$$

relevant symmetry global spin rotation

order parameter $\mathcal{O} = \sum_{x\in\Lambda_L} S_x^{(3)} \longrightarrow$ any direction

$$H_h = H - h\mathcal{O}$$

$$\langle \cdots \rangle_{\beta,h,L} := \frac{\int \prod_{x\in\Lambda_L} d\mathbf{S}_x \, (\cdots)\, e^{-\beta H_h}}{\int \prod_{x\in\Lambda_L} d\mathbf{S}_x \, e^{-\beta H_h}}$$

Theorem (Hohenberg 1967, Mermin–Wagner 1966)

$d = 1, 2$ for $\forall \beta < \infty$

$$\lim_{h\downarrow 0} \lim_{L\uparrow\infty} \frac{1}{L^d} \langle \mathcal{O} \rangle_{\beta,h,L} = 0$$

NO SSB

many proofs
easy

Theorem (Fröhlich–Simon–Spencer 1976)

$d \geq 3$ $\quad \exists \beta_0(d) > 0, \ \exists q_0(\beta) > 0$ for $\beta \geq \beta_0(d)$

$$\frac{1}{L^{2d}} \langle \mathcal{O}^2 \rangle_{\beta,0,L} \geq q_0(\beta) \quad \text{for } \forall \beta \geq \beta_0(d)$$

LRO

proof uses reflection positivity
↓
"BEC of spin waves"

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$$\begin{cases} m^*(\beta) = \lim_{h \downarrow 0} \lim_{L \uparrow \infty} \frac{1}{L^d} \langle \Theta \rangle_{\beta, h, L} \\ q(\beta) = \lim_{L \uparrow \infty} \frac{1}{L^{2d}} \langle \Theta^2 \rangle_{\beta, L} \end{cases}$$

argue that $m^*(\beta) \geq \sqrt{3 q(\beta)}$ *

When do we expect $m^*(\beta) = \sqrt{3 q(\beta)}$?

(need not be rigorous)

provable

⟨LRO and SSB in the ground state of quantum spin systems⟩

§ Some elementary linear algebra

- positive semidefinite operator (matrix)

  $\mathcal{H}$: a finite dim. Hilbertspace

  $\hat{A}$: hermitian operator on $\mathcal{H}$

$$\hat{A} \geq 0 \iff \langle \Phi, \hat{A}\Phi \rangle \geq 0 \text{ for } {}^\forall \Phi \in \mathcal{H}$$

$$\iff \text{min. e.v. of } \hat{A} \geq 0$$

$\hat{A}, \hat{B}$ hermitian $\quad \hat{A} - \hat{B} \geq 0 \iff \hat{A} \geq \hat{B}$

<u>Th.</u> $\hat{A} \geq 0,\ \hat{B} \geq 0 \Rightarrow \hat{A} + \hat{B} \geq 0$ (we don't assume $[\hat{A}, \hat{B}] = 0$)

$$\because \langle \Phi, (\hat{A} + \hat{B})\Phi \rangle \geq \langle \Phi, \hat{A}\Phi \rangle + \langle \Phi, \hat{B}\Phi \rangle \geq 0$$

<u>Corollary</u>. Let $\hat{H} = \sum_j \hat{H}_j$, and assume $\hat{H}_j \geq \epsilon_j$.
If Φ satisfies $\hat{H}_j \Phi = \epsilon_j \Phi$ for ${}^\forall j$ then Φ is a ground state of $\hat{H}$.

simultaneously minimizable ("frustration free")

This will be used repeatedly. (Too much)

- <u>Schwarz inequality</u>

(maximum exists)

$$|\langle \Phi, \hat{A}\hat{B}\Phi \rangle|^2 \leq \langle \Phi, \hat{A}^\dagger \hat{A}\Phi \rangle \langle \Phi, \hat{B}^\dagger \hat{B}\Phi \rangle \quad \text{for } {}^\forall \hat{A}, \hat{B}$$

- <u>Operator norm</u>

$$\|\hat{A}\| := \max_{\Phi \text{ s.t. } \|\Phi\| \neq 0} \frac{\|\hat{A}\Phi\|}{\|\Phi\|}, \quad \text{then } \|\hat{A}\hat{B}\| \leq \|\hat{A}\| \, \|\hat{B}\|$$

• Perron-Frobenius theorem

$n \times n$ matrix $A = (a_{ij})_{i,j=1,\dots,n}$

i) $a_{ij} \in \mathbb{R}$

ii) $a_{ij} \leq 0$ if $i \neq j$

iii) $\forall i \neq j$ are connected via nonvanishing elements of A

i.e. $\exists i_1, \dots, i_k$

s.t. $i_1 = i$, $i_k = j$, $a_{i_\ell i_{\ell+1}} \neq 0$ $(\ell = 1, \dots, k-1)$

<u>Theorem</u> Assume i), ii), iii), then $\exists$ a real nondegenerate e.v. λ_{PF} of A, and the corresponding eigenvector $v = (v_1, \dots, v_n)$ can be taken to satisfy $v_i > 0$. We have $\lambda_{PF} < \mathrm{Re}\,\lambda$ for any eigenvalue $\lambda \neq \lambda_{PF}$

(proof $\rightarrow$ see my book.)
elementary, but not easy

If A is real symmetric, λ_{PF} is the lowest eigenvalue
(ground state energy)

$\downarrow$

proof of the theorem is easy.

$\left(\begin{array}{l} v_i > 0 \\ \text{the g.s. wave function} \\ \text{is "nodeless"} \end{array} \right)$

§ Quantum spin systems — general definition and properties

- general lattice Λ
- spin $S = \frac{1}{2}, 1, \frac{3}{2}, \cdots$ fix

spin at site $x \in \Lambda$

$\mathcal{H}_x = \mathbb{C}^{2S+1}$ the Hibert space at x

$\hat{\mathbf{S}}_x = (\hat{S}_x^{(1)}, \hat{S}_x^{(2)}, \hat{S}_x^{(3)})$ spin operator at x

$$[\hat{S}_x^{(\alpha)}, \hat{S}_x^{(\beta)}] = i\sum_{\gamma} \epsilon_{\alpha\beta\gamma} \hat{S}_x^{(\gamma)}$$

$$(\hat{\mathbf{S}}_x)^2 = \sum_{\alpha=1}^{3} (\hat{S}_x^{(\alpha)})^2 = S(S+1)$$

$$\hat{S}_x^{\pm} := \hat{S}_x^{(1)} \pm i\, \hat{S}_x^{(2)}$$

basis states $\psi_x^{(\sigma)}$ $\quad \sigma = -S, -S+1, \cdots, S$

$$\begin{cases} \hat{S}_x^{(3)} \psi_x^{(\sigma)} = \sigma \psi_x^{(\sigma)} \\ \hat{S}_x^{\pm} \psi_x^{(\sigma)} = \sqrt{S(S+1) - \sigma(\sigma \pm 1)}\, \psi_x^{(\sigma \pm 1)} \end{cases}$$

quantum spin system on Λ

$$\mathcal{H} := \bigotimes_{x\in\Lambda} \mathcal{H}_x$$ # whole Hibert space

basis states $\Psi^{\sigma} := \bigotimes_{x\in\Lambda} \psi_x^{\sigma_x}$

spin config. $\sigma = (\sigma_x)_{x\in\Lambda}$, $\sigma_x = -S, -S+1, \cdots, S$

$\hat{S}_x^{(\alpha)}$ acts on $\psi_x^{\sigma_x}$

total spin $= (\hat{S}_{tot}^{(1)}, \hat{S}_{tot}^{(2)}, \hat{S}_{tot}^{(3)})$

$$\hat{\mathbf{S}}_{tot} := \sum_{x\in\Lambda} \hat{\mathbf{S}}_x$$

$$\hat{S}_{tot}^{\pm} := \hat{S}_{tot}^{(1)} \pm i \hat{S}_{tot}^{(2)}$$

The eigenvalues of $(\hat{\mathbf{S}}_{tot})^2$ is denoted as

$S_{tot}(S_{tot}+1)$

with $S_{tot} \in \{|\Lambda|S, |\Lambda|S-1, \cdots, \frac{1}{2} \text{ or } 0\}$

properties of $\mathbf{S}_x \cdot \mathbf{S}_y$ ← building block of the Heisenberg model

$\boxed{\forall S}$

(the most natural model for interacting spins → Part 3)

$$[\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y, \hat{S}_{tot}^{(\alpha)}] = 0 \qquad \alpha = 1, 2, 3$$

$$\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y = \frac{1}{2}\{(\hat{\mathbf{S}}_x + \hat{\mathbf{S}}_y)^2 - \hat{\mathbf{S}}_x^2 - \hat{\mathbf{S}}_y^2\}$$

$$= \frac{1}{2}\underbrace{(\hat{\mathbf{S}}_x + \hat{\mathbf{S}}_y)^2}_{} - S(S+1)$$

→ min. e.v. 0 non-deg.
max e.v. $2S(2S+1)$ $(4S+1)$-fold deg.

$\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y$
- min. e.v. $-S(S+1)$ non-deg. → singlet
- max e.v. S^2 $(4S+1)$ fold deg.

$$-S(S+1) \leq \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y \leq S^2$$

NOT symmetric

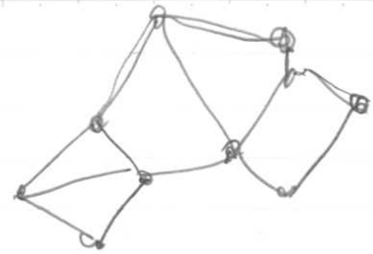
(classical vectors

$$-S^2 \leq \mathbf{S}_x \cdot \mathbf{S}_y \leq S^2 \text{)}$$

symmetric

DATE

§ Ferromagnetic Heisenberg model (warmup)



$(\Lambda, \mathcal{B})$ connected lattice

Λ: set of sites

$\mathcal{B}$: set of bonds $(x,y)=(y,x)$

$S=\frac{1}{2}, 1, \cdots$

Hamiltonian

$$\hat{H} = -\sum_{(x,y)\in\mathcal{B}} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y$$

then $[\hat{H}, \hat{S}^{(\alpha)}_{tot}] = 0$ $\quad \alpha = 1,2,3$

a ground state

Let $\Phi_\uparrow := \bigotimes_{x\in\Lambda} \psi^S_x$

Then $-\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y \Phi_\uparrow = -S^2 \Phi_\uparrow$ min. e.v. of $\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y$

$\therefore$ $\Phi_\uparrow$ is a ground state $\hat{H}\Phi_\uparrow = \underbrace{-|\mathcal{B}|S^2}_{= E_{GS}} \Phi_\uparrow$

$$\begin{cases} \dfrac{1}{|\Lambda|^2} \langle \Phi_\uparrow, (\hat{\mathbf{S}}_{tot})^2 \Phi_\uparrow \rangle = S^2 & \text{LRO} \\ \dfrac{1}{|\Lambda|} \langle \Phi_\uparrow, \hat{\mathbf{S}}_{tot} \Phi_\uparrow \rangle = (0,0,S) & \text{(spontaneous?) symmetry breaking} \end{cases}$$

other ground states

$$\Phi_\ell := \frac{(\hat{S}^-_{tot})^\ell \Phi_\uparrow}{\|(\hat{S}^-_{tot})^\ell \Phi_\uparrow\|} \qquad \ell = 0, 1, \cdots, 2|\Lambda|S$$

$$\hat{H}\Phi_\ell = E_{GS}\Phi_\ell$$

$2|\Lambda|S + 1$ ground states

connectedness is essential

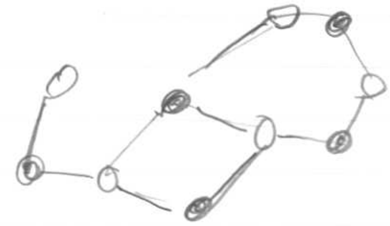
QS-1 Show that these are the only g.s.

(hints on Day 3.)

QS-2* Discuss LRO and SB for the whole space of g.s.

(I don't know the answer compette.)

§ Antiferromagnetic Heisenberg model
(often called Heisenberg AF)



$(\Lambda, \mathcal{B})$ connected, bipartite.

$\Lambda = A \cup B$ $\quad (x,y) \in \mathcal{B} \Rightarrow x \in A, y \in B$ or $x \in B, y \in A$

$S = \frac{1}{2}, 1, \ldots$

Hamiltonian

$$\hat{H} = \sum_{(x,y)\in\mathcal{B}} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y$$

spins want to point in the opposite direction

Néel state → ~~the~~ a ground state??

$$\Phi_{\text{Néel}} := \left(\bigotimes_{x \in A} \psi_x^{S} \right) \otimes \left(\bigotimes_{y \in B} \psi_y^{-S} \right)$$

noting that ~~recalling~~ $\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y = \hat{S}_x^{(3)} \hat{S}_y^{(3)} + \frac{1}{2}\left(\hat{S}_x^{+} \hat{S}_y^{-} + \hat{S}_x^{-} \hat{S}_y^{+} \right)$

$$\left(\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y \right)\left(\psi_x^{S} \otimes \psi_y^{-S} \right) = \underbrace{-S^2 \left(\psi_x^{S} \otimes \psi_y^{-S} \right)}_{\text{main if } S \gg 1 \text{ (classical)}} + S \left(\psi_x^{S-1} \otimes \psi_y^{-S+1} \right)$$

$\Phi_{\text{Néel}}$ is not a g.s. (uless $S = \infty$)

Theorem (Marshall 1955, Lieb-Mattis 1962)

Let $(\Lambda, \mathcal{B})$ be connected, bipartite with $|A|=|B|$.

Then the g.s. Φ_{GS} is unique and has $S_{tot}=0$.

It can be expanded as

$$\Phi_{GS} = \sum_{\substack{\sigma \\ (\sum_{x\in A}\sigma_x=0)}} C_\sigma \, (-1)^{\sum_{x\in A}(\sigma_x - S)} \, \Psi^\sigma$$

$$= \tilde{\Psi}^\sigma$$

with $C_\sigma > 0$.

proof Look for simultaneous eigenstates of $\hat{H}$, $\hat{S}^{(3)}_{tot}$, $(\hat{\mathbf{S}}_{tot})^2$.

Suppose $\hat{H}\Phi = E\Phi$, $\hat{S}^{(3)}_{tot}\Phi = M\Phi$ with $M \neq 0$

$\hookrightarrow$ so $S_{tot} \geq |M|$

then $\hat{H}\underbrace{(\hat{S}^-_{tot})^M \Phi}_{\text{nonvanishing}} = (\hat{S}^-_{tot})^M H \Phi$

$= E(\hat{S}^-_{tot})^M \Phi$.

We can find all the energy eigenvalues in the subspace with $\hat{S}^{(3)}_{tot}=0$.

basis $\tilde{\Psi}^\sigma$ with $\sum_x \sigma_x = 0$

then (i) $\langle \tilde{\Psi}^\sigma, \hat{H}\tilde{\Psi}^{\sigma'} \rangle \in \mathbb{R}$.

(ii) $\langle \tilde{\Psi}^\sigma, \hat{H}\tilde{\Psi}^{\sigma'} \rangle \leq 0$ if $\sigma \neq \sigma'$

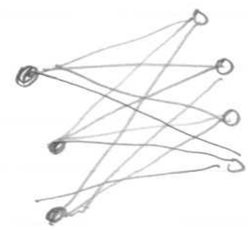
(iii) $\forall \sigma, \sigma'$ with $\sum \sigma_x = \sum \sigma'_x = 0$ are connected via $\hat{H}$.

the PF theorem implies that

the g.s. within the subspace is unique, and $C_\sigma > 0$.

Φ_{GS}

What is S_{tot} for Φ_{GS} ?



toy model on the same lattice

$$\hat{H}_{toy} = \left(\sum_{x\in A}{}' \hat{\$}_x\right)\cdot\left(\sum_{y\in B}{}' \hat{\$}_y\right)$$

$$= \frac{1}{2}\Big\{\underbrace{(\hat{\$}_{tot})^2}_{=0} - \underbrace{(\hat{\$}_A)^2}_{= S|A|(S|A|+1)} - \underbrace{(\hat{\$}_B)^2}_{= S|B|(S|B|+1)}\Big\}$$

We get the g.s. when

$$\therefore (\hat{\$}_{tot})^2\, \Phi_{toyGS} = 0$$

We also have $\Phi_{toyGS} = \sum_{\substack{\sigma \\ (\Sigma \sigma_x = 0)}}{}' C'(\sigma)\, \tilde{\Psi}^{\sigma}$ with $C'(\sigma) > 0$

$$\therefore \langle \Phi_{GS}, \Phi_{toyGS} \rangle \neq 0$$

$$\Downarrow$$

$$(\hat{\$}_{tot})^2\, \Phi_{GS} = 0$$

The G.S. has $S_{tot} = 0$ and hence is unique //

QS-3 extend the theorem to the case with $|A| \neq |B|$

—— o ——

The nature of Φ_{GS} ? $\longrightarrow$ depends on $(\Lambda, \mathcal{B})$.

$d \geq 2$ today $\quad d = 1$ day 2.

§ LRO in the ground state ~~for~~ of the Heisenberg AF in $d \geq 2$

Λ_L : $L \times \cdots \times L$ d-dim. hypercubic lattice.

$\mathcal{B}_L$: the set of n.n. bonds (periodic b.c.)

$\Lambda_L = A \cup B$ with

$$A = \{x = (x_1, \cdots, x_d) \in \Lambda_L \mid \textstyle\sum_i x_i \text{ even}\}$$

$$B = \{x = (x_1, \cdots, x_d) \in \Lambda_L \mid \textstyle\sum x_i \text{ odd}\}$$

Hamiltonian.

$$\hat{H} := \sum_{(x,y) \in \mathcal{B}_L} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y$$

Symmetry

global spin rotation

$$\hat{U} = \exp\left[i\theta \sum_{x \in \Lambda_L} \hat{S}^{(\alpha)}_x\right]$$

AF order parameter

$$\hat{\mathcal{O}}^{(\alpha)} := \sum_{x \in \Lambda_L} (-1)^x \hat{S}^{(\alpha)}_x \qquad \alpha = 1, 2, 3$$

$$(-1)^x = \begin{cases} 1 & x \in A \\ -1 & x \in B \end{cases}$$

<u>Theorem</u> $d \geq 3$, $\forall S$, $\quad d \geq 2$, $S \geq 1$. $\exists q_0 > 0$ s.t. (depends on d, S but not on L)

$$\frac{1}{L^{2d}} \langle \Phi_{GS}, (\hat{\mathcal{O}}^{(\alpha)})^2 \Phi_{GS} \rangle \geq q_0 \quad \text{for } \forall L, \ \alpha = 1, 2, 3$$

(proof uses reflection positivity due to Dyson-Lieb-Simon 1978

Neves-Perez 1986, Kennedy-Lieb-Shastry 1988, Kubo-Kishi 1988, ...)

Thus.

$$(-1)^{x-y}\langle \Phi_{GS}, \hat{S}_x \cdot \hat{S}_y \Phi_{GS} \rangle \gtrsim 3q_0$$

for $\forall x, y$

long-range AF order (or Néel order)

But the uniqueness implies

$$\langle \Phi_{GS}, \hat{O}^{(\alpha)} \Phi_{GS} \rangle = 0 \quad \text{for } \alpha = 1,2,3$$

NO SSB

"LRO without SSB" is common in the g.s. of quantum many-body systems where the Hamiltonian and the order parameter do not commute.

(magnetism)

superconductivity

Bose-Einstein cond.

the simplest example

$\Downarrow$

$\Downarrow$

§ Ising model under transverse magnetic field

$$\Lambda = \{1, 2, \cdots, L\} \qquad S = \frac{1}{2}$$

$$\hat{H} = -\sum_{x=1}^{L-1} \hat{S}^{(3)}_x \hat{S}^{(3)}_{x+1} - \delta \sum_{x=1}^{L} \hat{S}^{(1)}_x \qquad (\delta \geq 0)$$

open b.c.

ferromagnetic LRO and SB

$\underline{\delta = 0}$ (Ising ferro)

two g.s. $\Phi_\uparrow = \bigotimes_{x=1}^{L} \psi^\uparrow_x$, $\Phi_\downarrow = \bigotimes_{x=1}^{L} \psi^\downarrow_x$

$$E^0_{GS} = -\frac{L-1}{4}$$

1st excited st.

$\uparrow\uparrow\downarrow\downarrow\downarrow$ (y)

$$\Theta_y := \left(\bigotimes_{x=1}^{y} \psi^\uparrow_x\right) \otimes \left(\bigotimes_{x=y+1}^{L} \psi^\downarrow_x\right) \qquad y = 1, \cdots, L-1$$

$$\tilde{\Theta}_y := \left(\bigotimes_{x=1}^{y} \psi^\downarrow_x\right) \otimes \left(\bigotimes_{x=y+1}^{L} \psi^\uparrow_x\right)$$

$$E^0_{1st} = E^0_{GS} + \frac{1}{2}$$

$\underline{0 < \delta \ll 1}$

the exact g.s.

$$\Phi_{GS} \simeq \frac{1}{\sqrt{2}}(\Phi_\uparrow + \Phi_\downarrow)$$

CATS!

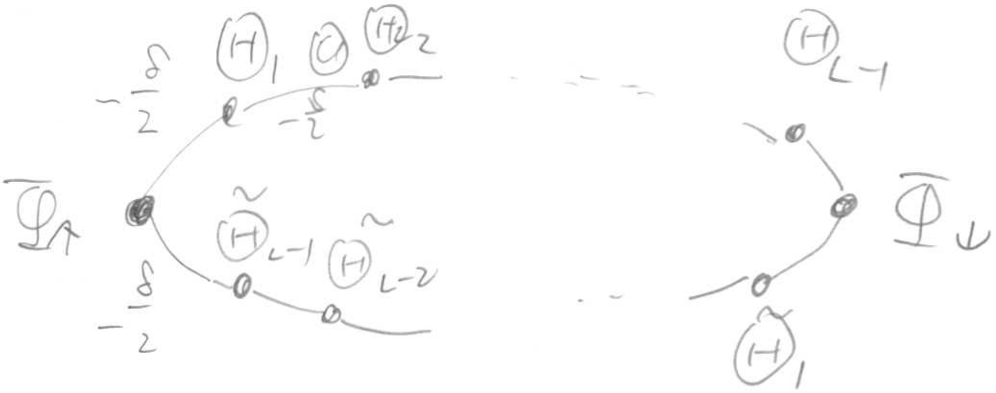
$$\Phi_{1st} \simeq \frac{1}{\sqrt{2}}(\Phi_\uparrow - \Phi_\downarrow)$$

$$E_{1st} - E_{GS} \simeq \delta^L$$

from exact solution

low-lying excited state

QS-4 Show this (nonrigorously)
by analyzing the low-energy effective model ~~in~~
with $\hat{H}$ in the subspace spanned by $\Phi_\uparrow, \Phi_\downarrow, \Theta_y, \tilde{\Theta}_y$



order parameter $\hat{O} = \hat{S}^{(3)}_{tot.}$

$$\left(\hat{O}\,\Phi_\uparrow = \frac{L}{2}\Phi_\uparrow \quad \hat{O}\,\Phi_\downarrow = -\frac{L}{2}\Phi_\downarrow \right)$$

$$\therefore \begin{cases} \langle \Phi_{GS}, \hat{O}^2 \Phi_{GS} \rangle \simeq \dfrac{L^2}{4} & \text{LRO} \\ \langle \Phi_{GS}, \hat{O}\, \Phi_{GS} \rangle \simeq 0 & \text{without SSB} \end{cases}$$

Φ_{GS} : exact g.s. for finite L but unphysical.

$\Downarrow$

$\left(\frac{\hat{O}}{L}\right)$ fluctuates!

Physically natural "g.s." are $\Phi_\uparrow, \Phi_\downarrow$

$$\begin{cases} \langle \Phi_\uparrow, \hat{O}^2 \Phi_\uparrow \rangle \simeq \dfrac{L^2}{4} & \text{LRO} \\ \langle \Phi_\uparrow, \hat{O}\, \Phi_\uparrow \rangle \simeq \dfrac{L}{2} & \text{SSB} \end{cases}$$

$$\sqrt{\left\langle \Phi_\uparrow, \left(\frac{\hat{O}}{L}\right)^2 \Phi_\uparrow \right\rangle - \left(\left\langle \Phi_\uparrow, \frac{\hat{O}}{L} \Phi_\uparrow \right\rangle \right)^2} \xrightarrow[L\uparrow\infty]{} 0$$

$\hat{O}/L$ does not fluctuate!

$$\begin{cases} \Phi_\uparrow \simeq \dfrac{1}{\sqrt{2}}(\Phi_{GS} + \Phi_{1st}) \\ \Phi_\downarrow \simeq \dfrac{1}{\sqrt{2}}(\Phi_{GS} - \Phi_{1st}) \end{cases}$$

physical "g.s." are linear combinations of the exact g.s. and the low-lying excited state.

$$\left(\text{also note } \Phi_{1st} \simeq \frac{\hat{O}\,\Phi_{GS}}{\|\hat{O}\,\Phi_{GS}\|} \longrightarrow \text{Horsch-von der Linden} \right)$$

§ From LRO to SSB   Kaplan - Horsch - von der Linden.

consider
- Ising under trans. field
- Heisenberg AF on $\Lambda_L \subset \mathbb{Z}^d$

or

more general models on Λ_L

$\hat{\Theta}^\dagger = \hat{\Theta}$ is crucial

$$\hat{\Theta} = \begin{cases} \hat{S}^{(3)}_{tot} & \text{Ising} \\ \hat{\Theta}^{(\alpha)} = \sum_x (-1)^x \hat{S}^{(\alpha)}_x & \text{Heisenberg AF} \end{cases}$$

assume $\langle \Phi_{GS}, \hat{\Theta}^2 \Phi_{GS} \rangle \geq q_0 L^{2d}$ LRO

$\langle \Phi_{GS}, \hat{\Theta}^n \Phi_{GS} \rangle = 0 \quad (n = 1, 3.)$ NO SSB

construction of low-lying excited state   Horsch - von der Linden 1988

trial state $\Gamma = \dfrac{\hat{\Theta} \Phi_{GS}}{\| \hat{\Theta} \Phi_{GS} \|}$, $\langle \Phi_{GS}, \Gamma \rangle = 0$

$$\langle \Gamma, \hat{H} \Gamma \rangle - E_{GS}$$

$$= \frac{\langle \Phi_{GS}, \hat{\Theta} \hat{H} \hat{\Theta} \Phi_{GS} \rangle - \frac{1}{2} \langle \Phi_{GS}, \hat{\Theta}^2 \hat{H} \Phi_{GS} \rangle - \frac{1}{2} \langle \Phi_{GS}, \hat{H} \hat{\Theta}^2 \Phi_{GS} \rangle}{\langle \Phi_{GS}, \hat{\Theta}^2 \Phi_{GS} \rangle}$$

$$= \frac{\langle \Phi_{GS}, [\hat{\Theta}, [\hat{H}, \hat{\Theta}]] \Phi_{GS} \rangle}{2 \langle \Phi_{GS}, \hat{\Theta}^2 \Phi_{GS} \rangle}$$

now $[\hat{H}, \hat{\Theta}] = \sum_x \bigcirc_x$ ← local around x

$$[\hat{\Theta}, [\hat{H}, \hat{\Theta}]] = \sum_x \bigcirc_x$$

$$\therefore \| [\hat{\Theta}, [\hat{H}, \hat{\Theta}]] \| \leq \text{const}\, L^d$$

$$\therefore \quad 0 \leq \langle \Gamma, \hat{H} \Gamma \rangle - E_{GS} \leq \frac{\text{const}\, L^d}{2 q_0 L^{2d}} = C L^{-d}$$

Theorem $E_{1st} \leq E_{GS} + C L^{-d}$

(LRO without SSB ⟶ $\exists$ low-lying excited state)

Low-lying states with SSB

$\Xi = \frac{1}{\sqrt{2}} (\Phi_{GS} + \Gamma)$, $\quad \langle \Xi, \hat{H} \Xi \rangle \leq E_{GS} + \frac{C}{2} L^{-d}$

low-lying state

$$\langle \Xi, \hat{\Theta} \Xi \rangle = \frac{1}{2} \left\langle \left(\Phi_{GS} + \frac{\hat{\Theta} \Phi_{GS}}{\| \hat{\Theta} \Phi_{GS} \|} \right), \left(\hat{\Theta} \Phi_{GS} + \frac{\hat{\Theta}^2 \Phi_{GS}}{\| \hat{\Theta} \Phi_{GS} \|} \right) \right\rangle$$

$$= \frac{\langle \Phi_{GS}, \hat{\Theta}^2 \Phi_{GS} \rangle}{\| \hat{\Theta} \Phi_{GS} \|} = \sqrt{\langle \Phi_{GS}, \hat{\Theta}^2 \Phi_{GS} \rangle}$$

$$\geq \sqrt{q_0}\, L^d$$

(Ξ is a low-lying state with SSB)

so is $\frac{1}{\sqrt{2}} (\Phi_{GS} - \Gamma)$

SSB under "infinitessimally small external field"

Hamiltonian with (staggered) magnetic field

$$\hat{H}_h = \hat{H} - h\hat{\mathcal{O}}, \qquad h>0$$

$\Phi_{GS,h}$ the GS of H_h

Obviously

$$\langle \Xi, \underbrace{\hat{H}_h}_{\hat{H}-h\hat{\mathcal{O}}} \Xi \rangle \geq \langle \Phi_{GS,h}, \underbrace{\hat{H}_h}_{\hat{H}-h\hat{\mathcal{O}}} \Phi_{GS,h} \rangle$$

devide by $h L^d$

$$\frac{1}{L^d}\langle \Phi_{GS,h}, \hat{\mathcal{O}}\Phi_{GS,h}\rangle \geq \frac{1}{L^d}\langle \Xi, \hat{\mathcal{O}}\Xi\rangle + \frac{1}{hL^d}\left\{\langle \Phi_{GS,h}, \hat{H}\Phi_{GS,h}\rangle - \langle \Xi, \hat{H}\Xi\rangle\right\}$$

$$\geq \sqrt{q_0} + \frac{1}{hL^d}\left\{\underbrace{E_{GS} - \langle \Xi, \hat{H}\Xi\rangle}_{"O(L^{-d})"}\right\}$$

the gs energy with $h=0$

Theorem (Kaplan–Horsch–von der Linden, 1989)

$$\lim_{h\downarrow 0}\lim_{L\uparrow\infty}\frac{1}{L^d}\langle \Phi_{GS,h}, \hat{\mathcal{O}}\Phi_{GS,h}\rangle \geq \sqrt{q_0}$$

$$\boxed{\underset{(h=0)}{\text{LRO}} \rightarrow \underset{+\,\text{LRO}}{\text{SSB}}}\ (h\downarrow 0)$$ for quite general quantum many-body systems

NOT YET THE WHOLE STORY!

infinitely many "g.s. with SSB" many low-lying states?? * Yes

§ From LRO to SSB in systems with a continuos symmetry

— Koma-Tasaki theorems 1994

Heisenberg AF on $\Lambda_L \subset \mathbb{Z}^d$ (other models with SU(2) or U(1) symmetry)

$$\hat{\mathcal{O}}^{\pm} := \sum_{x\in\Lambda_L}{}' (-1)^x \hat{S}_x^{\pm}, \quad \hat{\mathcal{O}}^{(\alpha)} = \sum_{x\in\Lambda_L}{}' (-1)^x \hat{S}_x^{(\alpha)}$$

$$\hat{S}_{\mathrm{Tot}}^{(3)} \Phi_{GS} = 0, \quad \hat{H}\Phi_{GS} = E_{GS}\Phi_{GS}$$

Φ_{GS} unique g.s.

$$\boxed{\langle \Phi_{GS}, (\hat{\mathcal{O}}^{(\alpha)})^2 \Phi_{GS} \rangle \geq q_0 L^{2d}} \quad (\alpha = 1,2,3)$$

LRO

for $M = 1, 2, \cdots$

$$\Gamma_M := \frac{(\hat{\mathcal{O}}^{+})^M \Phi_{GS}}{\| (\hat{\mathcal{O}}^{+})^M \Phi_{GS} \|} \qquad \Gamma_{-M} := \frac{(\hat{\mathcal{O}}^{-})^M \Phi_{GS}}{\| (\hat{\mathcal{O}}^{-})^M \Phi_{GS} \|}$$

<u>Theorem</u> For $\forall M$ s.t. $|M| \leq \text{const.}\, L^{d/2}$

$$\left| \langle \Gamma_M, \hat{H}\Gamma_M \rangle - E_{GS} \right| \leq \text{const}\, \frac{M^2}{L^d}$$

(proof: not easy)

well-known in the numerical community

Since $\hat{S}_{\mathrm{Tot}}^{(3)} \Gamma_M = M \Gamma_M$,

"Anderson's tower"

$\exists \Phi_M$ s.t.

$$\hat{S}_{\mathrm{Tot}}^{(3)} \Phi_M = M \Phi_M$$

$$\hat{H}\Phi_M = E_M \Phi_M \text{ with } E_M \leq E_{GS} + \text{const}\, \frac{M^2}{L^d}$$

There are <u>ever increasing</u> series of low-lying excited states.

$d=3$ exc. en. $\sim \frac{1}{L^3}$ (spin-wave exc. $\propto \frac{1}{L^2}$)

lowlying states with (full) SSB

$$\Xi_k := \frac{1}{\sqrt{2k+1}} \left\{ \Phi_{GS} + \sum_{M=1}^{k} \left(\Gamma_M + \Gamma_{-M} \right) \right\}$$

Néel order

Theorem
$$\lim_{k\uparrow\infty} \lim_{L\uparrow\infty} \frac{1}{L^d} \langle \Xi_k, \hat{O}^{(1)} \Xi_k \rangle \geq \sqrt{3}\, q_0$$

SSB

(proof: rather techical)

Rem. Of course $\lim_{k\uparrow\infty} \lim_{L\uparrow\infty} \langle \Xi_k, (\hat{O}^{(1)})^2 \Xi_k \rangle \geq 3 q_0$ LRO

$\left(\because \langle \hat{O}^2 \rangle \geq (\langle \hat{O} \rangle)^2 \right.$ in general $\left.\right)$

Rem, Horsch-von der Linden state gives

$$\lim_{L\uparrow\infty} \frac{1}{L^d} \langle \Xi, \hat{O}^{(1)} \Xi \rangle \geq \sqrt{q_0} \leftarrow \text{NOT full SB}$$

Corollary Let $\Phi_{GS,h}$ be the g.s. of $\hat{H} - h\hat{O}^{(1)}$

$$\lim_{h\downarrow 0} \lim_{L\uparrow\infty} \frac{1}{L^d} \langle \Phi_{GS,h}, \hat{O}^{(1)} \Phi_{GS,h} \rangle \geq \sqrt{3}\, q_0$$

optimal
($\sqrt{2}$ if SO(2) symmetry)

with LRO AND SSB

> Physically natural g.s. are linear combinations of ever increasing numbers of low-lying states as in Ξ_k.

§ Ground states of infinite systems

Heisenberg AF on Λ_L, assume $\exists$ LRO in Φ_{GS} the unique g.s.

<u>algebra of operators</u>

$$\tilde{\mathfrak{A}} = \{ \text{polynomials of } \hat{S}_x^{(\alpha)},\ x \in \mathbb{Z}^d,\ \alpha = 1, 2, 3 \}$$

\mathfrak{A}

$$\omega_0(\hat{A}) := \lim_{L \uparrow \infty} \langle \Phi_{GS}, \hat{A}\, \Phi_{GS} \rangle$$

Ω : solid angle. $\hat{U}_\Omega$ a suitable rotation $(1,0,0) \to \Omega$

$$\omega_\Omega(\hat{A}) := \lim_{k \uparrow \infty} \lim_{L \uparrow \infty} \langle \hat{U}_\Omega \Xi_k, \hat{A}\, \hat{U}_\Omega \Xi_k \rangle$$

<u>Theorem</u> (Koma Tasaki)

$\omega_0(\cdot)$ and $\omega_\Omega(\cdot)$ are g.s. infinitely many g.s. !

$$\left(\begin{array}{l} \text{i.e., for } \forall (x,y) \text{ s.t. } |x-y| = 1 \\ \omega_0(\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y) = \omega_\Omega(\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y) = \epsilon_{GS} := \lim_{L \uparrow \infty} \dfrac{E_{GS,L}}{|\mathcal{B}_L|} \end{array} \right)$$

$$\omega_0(\hat{S}_x^{(\alpha)}) = 0 \qquad \alpha = 1, 2, 3$$

$$(-1)^x\, \omega_\Omega(\Omega \cdot \hat{\mathbf{S}}_x) \geq \sqrt{3}\, q_0$$

$$\omega_\Omega(\mathbf{v} \cdot \hat{\mathbf{S}}_x) = 0 \quad \text{if } \mathbf{v} \cdot \Omega = 0$$

and

$$\omega_0(\cdot) = \frac{1}{4\pi} \int d\Omega\, \omega_\Omega(\cdot)$$

//

$\omega_0(\cdot)$ is ~~not ergodic~~ unphysical $\left(\frac{\hat{O}_L^{(d)}}{L^d}\right)$ has big fluctuation

Conjecture

$\omega_\Omega(\cdot)$ is ergodic (physical state)

($\forall$ macroscopic quantities has small fluctuation in $\omega_\Omega(\cdot)$)

Then

$$\omega_0(\cdot) = \frac{1}{4\pi}\overbrace{\int d\Omega\, \omega_\Omega(\cdot)}^{\text{mathematically natural decomposition into ergodic states}}$$

$\omega_0(\cdot)$ → unphysical g.s. $\boxed{\text{LRO without SSB}}$

(obtained from the unique g.s. Φ_{GS})

$\omega_\Omega(\cdot)$ → physical g.s. with Néel order

in reality one of $\omega_\Omega(\cdot)$ is selected (by some reasons)

$\boxed{\text{SSB}}$

how??

§ equilibrium (remarks)

Heisenberg model on Λ_L

$d=1,2$ no LRO or SSB if $T \gneqq 0$. (Hohenberg, 1967; Mermin-Wagner, 1966)

ferro or AF

$d \geq 3$ AF LRO at suff. low temperatures

(Dyson-Lieb-Simon 1978
Kennedy-Lieb-Shastry 1988)

BEC of spin waves

SSB (Koma-Tasaki 1993)

↓

$\hat{H}$ and $\hat{\Theta}$ "almost commute" for large L

↓

extension of the Griffiths' theorem

"physics" may not be very different from classical situation

no results for Heisenberg ferro!

⟨LRO and SSB associated with Bose-Einstein condensation⟩

§ Bosons on a lattice (optical lattice)

Λ_L , $\mathcal{B}_L$

$\begin{cases} \hat{a}_x & \text{annihilation operator of a boson at } x \in \Lambda_L \\ \hat{a}_x^\dagger & \text{creation} \quad \text{"} \end{cases}$

$$[\hat{a}_x, \hat{a}_y^\dagger] = \delta_{x,y} \quad \text{for } \forall x, y \in \Lambda_L$$

Φ_{vac} unique state s.t. $\hat{a}_x \Phi_{vac} = 0$ for $\forall x$

↳ state with no particles on Λ.

N: particle number

- Hilbert space spanned by $a_{x_1}^\dagger a_{x_2}^\dagger \cdots a_{x_N}^\dagger \Phi_{vac}$ with any $x_1, x_2, \ldots, x_N \in \Lambda_L$

- fix $\rho = \dfrac{N}{L^d}$ and change L, N.

$$\hat{H} = \overbrace{-t \sum_{(x,y)\in\mathcal{B}_L}' (\hat{a}_x^\dagger \hat{a}_y + \hat{a}_y^\dagger a_x)}^{\text{hopping}} + \overbrace{g \sum_{x\in\Lambda_L}' \hat{n}_x(\hat{n}_x - 1)}^{\text{interaction}}$$

$\hat{n}_x = \hat{a}_x^\dagger \hat{a}_x$

$t > 0 \qquad g \geq 0$

§ off-diagonal LRO

<u>relevant symmetry</u> U(1) gauge symmetry $U(\theta) = e^{i\theta\hat{N}}$

$\hat{N} = \sum'_{x \in \Lambda_L} \hat{n}_x$

order parameters

$$\hat{\mathcal{O}} = \sum'_{x \in \Lambda_L} \frac{\hat{a}_x^{\dagger} + \hat{a}_x}{2} \quad \left(\text{or } \hat{\mathcal{O}}^{+} = \sum'_{x} \hat{a}_x^{\dagger}, \ \hat{\mathcal{O}}^{-} = \sum'_{x} \hat{a}_x \right)$$

For a wide range of g/t, ρ, it is expected that there is BEC in the sense that

$$\langle \Phi_{GS}, \hat{\mathcal{O}}^2 \Phi_{GS} \rangle \geq q_0 L^{2d} \quad \text{for } \forall L$$

$$(q_0 > 0)$$

$$\Downarrow$$

$$\langle \Phi_{GS}, a_x^{\dagger} a_y \Phi_{GS} \rangle \gtrsim 2 q_0 \quad \text{for } \forall x, y$$

off-diagonal LRO

N particles!

BUT CLEARLY $\langle \Phi_{GS}, \hat{\mathcal{O}} \Phi_{GS} \rangle = 0$ NO SSB

This has been proved only when $\rho = \frac{1}{2}$, $g/t = \infty$

(Kubo-Kishi, Kennedy-Lieb-Shastry)
1988 1988

mapping to a quantum spin system

$$\hat{a}^\dagger_x \longleftrightarrow \hat{S}^+_x$$

$$\hat{a}_x \longleftrightarrow \hat{S}^-_x$$

$$\hat{\Theta} \longleftrightarrow \hat{S}^{(1)}_{tot}$$

$$\frac{N}{L^d} = \frac{1}{2} \longleftrightarrow \hat{S}^{(3)}_{tot} = 0$$

Assume $N = \frac{L^d}{2}$, and $^\exists$LRO

→ assume $N=\frac{L^d}{2}$, $\exists$ LRO

§ Low-lying states with explicit symmetry breaking

use Koma-Tasaki construction of low-lying states (extra phase)

$$\Xi_{k,\theta} := \frac{1}{\sqrt{2k+1}}\left\{\Phi_{GS} + \sum_{M=1}^{k}\left(\frac{e^{-i\theta M}(\hat{O}^+)^M \Phi_{GS}}{\|(\hat{O}^+)^M\Phi_{GS}\|} + \frac{e^{i\theta M}(\hat{O}^-)^M\Phi_{GS}}{\|(\hat{O}^-)^M\Phi_{GS}\|}\right)\right\}$$

$0 \le \theta \le 2\pi$

Then
$$\lim_{k\uparrow\infty}\lim_{L\uparrow\infty}\frac{1}{L^d}\langle \Xi_{k,\theta}, \hat{O}^+ \Xi_{k,\theta}\rangle e^{-i\theta} \geq \sqrt{2q_0}$$

$$\lim_{k\uparrow\infty}\lim_{L\uparrow\infty}\frac{1}{L^d}\langle \Xi_{k,\theta}, \hat{O}^- \Xi_{k,\theta}\rangle e^{i\theta} \geq \sqrt{2q_0}$$

Creation and annillation operators have nonvanishing expectation values

Hamiltonian with external field $(h \ge 0)$

$$\hat{H}_{\theta,h} := \hat{H} - h(e^{-i\theta}\hat{O}^+ + e^{i\theta}\hat{O}^-)$$

$\xrightarrow{GS} \Phi_{GS,\theta,h}$

$$\lim_{h\downarrow 0}\lim_{L\uparrow\infty}\frac{1}{L^d}\langle \Phi_{GS,\theta,h}, \hat{O}^+ \Phi_{GS,\theta,h}\rangle e^{-i\theta} \geq \sqrt{2q_0}$$

SSB associated with ODLRO

Φ_{GS} inf. vol. g.s.

$\Xi_{k,\theta}$

$$\omega_0(\cdot) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \omega_\theta(\cdot)$$

ODLRO without U(1) SSB

U(1) SSB + ODLRO

BUT.

- one can never generate a field like

$$-e^{-i\theta}\sum_x \hat{a}^\dagger_x - e^{i\theta}\sum_x \hat{a}_x$$

- superpositions of states with different particle numbers are meaningless.

$$\Phi_N + \Phi_{N+1} \; ?$$

particle number conservation

$$\Phi_N \otimes \textcircled{H}_{n-N} + \Phi_{N+1} \otimes \textcircled{H}_{n-N-1}$$

outside

$$\omega_0 = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; \omega_\theta$$

$\boxed{\text{physically realizable g.s}}$

fictitious states which are "natural" from theoretical point of view

$\downarrow$

mean-field theory

(such states are realistic for photons)

<u>ergodicity?</u>

any observables ω_θ ergodic ω_0 non-ergodic

gauge inv. observables <u>ω_0 is ergodic</u>

§ Physical "SSB" in a coupled system

$\left(\text{Recall that } \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \Xi_{k,\theta} = \frac{1}{\sqrt{2k+1}} \Phi_{GS} \right)$

Prepare two identical systems and consider

$$\Gamma_\varphi = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \Xi_{k,\theta} \otimes \Xi_{k,\theta+\varphi}$$

$$= \frac{1}{2k+1} \sum_{M=-k}^{k} e^{iM\varphi} \underbrace{\frac{(\hat{O}^{+})^{M}\Phi_{GS}}{\|(\hat{O}^{+})^{M}\Phi_{GS}\|} \otimes \frac{(\hat{O}^{+})^{-M}\Phi_{GS}}{\|(\hat{O}_{+})^{-M}\Phi_{GS}\|}}$$

states with a definite particle number $2N$

the two systems have a definite relative phase φ

$$\hat{H} = \hat{H}\otimes \mathbb{1} + \mathbb{1}\otimes\hat{H} - \varepsilon \sum_{\substack{x\in\Lambda_L \\ y\in\Lambda'_L}} \{ e^{-i\varphi}\hat{a}_x \hat{a}_y^{\dagger} + e^{i\varphi}\hat{a}_x^{\dagger}\hat{a}_y \}$$

$\to \Phi_{GS,\theta,\varepsilon}$

<u>Theorem</u>

$$\lim_{\varepsilon\downarrow 0}\lim_{L\uparrow\infty} \frac{1}{L^{2d}} \left\langle \Phi_{GS,\theta,\varepsilon}, \left(\sum_{x\in\Lambda_L}\hat{a}_x^{\dagger}\right)\left(\sum_{y\in\Lambda'_L}\hat{a}_y\right) \Phi_{GS,\theta,\varepsilon} \right\rangle e^{i\varphi}$$

$\geq \text{const } q_0$

?

a kind of SSB for the relative phase

BEC-1***

Prove the theorem with the optimal constant,

and publish!

(acknowledge me)

< No go theorems for "time crystal" >

§ time crystal ?

AF order $\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow$

translation invariance is also broken

crystal  SSB of translation invariance

"time crystal" Wilczek 2012

Can there be a SSB in temporal direction ?

$$\frac{1}{L^d} \langle \Phi_{\text{"GS"}}, \hat{A}(t)\, \Phi_{\text{"GS"}} \rangle$$ oscilates in time for large L.

bulk quatity

exact g.s.

Obviously $\frac{d}{dt} \langle \Phi_{GS}, \hat{A}(t)\, \Phi_{GS} \rangle = 0$

but this does not rule out "spontaneaus" oscilation.

quantum spin system on Λ_L

$$\mathcal{N}_x = \{ y \in \Lambda_L \mid |x-y| \le \ell \}$$

fixed

$$\hat{H}_L = \sum_{x \in \Lambda_L}{}' \hat{h}_x \quad , \quad \hat{h}_x \text{ acts only on } \mathcal{N}_x$$

$$\|\hat{h}_x\| \le h_0$$

$$\hat{A}_L = \sum_{x \in \Lambda_L}{}' \hat{a}_x \quad , \quad \hat{a}_x \text{ acts only on } \mathcal{N}_x ,$$

$$\|\hat{a}_x\| \le a_0$$

hermitian

§ Absence of LRO

LRO for time-crystal.

Oscilation of $\frac{1}{L^{2d}} \langle \Phi_{GS}, \hat{A}_L(t) \hat{A}_L(0) \Phi_{GS} \rangle$ ← exact gs.

<u>Theorem</u> (Watanabe-Oshikawa 2014)

$$\frac{1}{L^{2d}} \left| \langle \Phi_{GS}, \hat{A}_L(t) \hat{A}_L(0) \Phi_{GS} \rangle - \langle \Phi_{GS}, \hat{A}_L(0) \hat{A}_L(0) \Phi_{GS} \rangle \right| \leq \text{const.} \frac{|t|}{L^d} \text{ for } {}^\forall t.$$

$$\therefore \lim_{L \uparrow \infty} \frac{1}{L^{2d}} \langle \Phi_{GS}, \hat{A}_L(t) \hat{A}_L(0) \Phi_{GS} \rangle$$

is indep. of t.

NO LRO corresponding to a "time crystal".

<u>proof</u>

$$\langle \Phi_{GS}, \hat{A} e^{-i\hat{H}t} \hat{A} \Phi_{GS}\rangle e^{iE_{GS}t}$$

$$\langle \Phi_{GS}, e^{i\hat{H}t} \hat{A} e^{-i\hat{H}t} A \Phi_{GS}\rangle - \langle \Phi_{GS}, \hat{A}^2 \Phi_{GS}\rangle$$

$$= \int_0^t ds \frac{d}{ds}\left\{\langle \Phi_{GS}, \hat{A} e^{-i\hat{H}s} \hat{A} \Phi_{GS}\rangle e^{iE_{GS}s}\right\}$$

$$= -i \int_0^t ds \langle \Phi_{GS}, \hat{A} (\hat{H} - E_{GS}) e^{-i\hat{H}s} \hat{A} \Phi_{GS}\rangle e^{iE_{GS}s}$$

absolute value

$$\left|\langle \Phi_{GS}, \hat{A} \sqrt{\hat{H} - E_{GS}}\, e^{-i\hat{H}s} \sqrt{\hat{H} - E_{GS}}\, \hat{A} \Phi_{GS}\rangle\right|$$

$$\overset{\text{Schwarz}}{\leq} \langle \Phi_{GS}, \hat{A} (\hat{H} - E_{GS}) \hat{A} \Phi_{GS}\rangle$$

$$= \frac{1}{2}\langle \Phi_{GS}, [\hat{A}, [\hat{H}, \hat{A}]] \Phi_{GS}\rangle \leq \text{const } L^d$$

$\| \cdots \| \leq \text{const } L^d$

Horsch-von der Liden

<u>picture</u> To see a bulk oscilation we need

$$\langle \Phi_{GS}, \hat{A}^2 \Phi_{GS}\rangle \sim O(L^{2d}) \qquad \text{LRO}$$

But this means $\dfrac{\hat{A}\Phi_{GS}}{\|\hat{A}\Phi_{GS}\|}$ is a low-lying exc.

$\Downarrow$

slow oscilation

TC-1 Examine the W-O theorem for models with long-range interaction (including the mean-field model) with

$$\hat{H} = \sum_{x,y} \hat{h}_{x,y}, \qquad \sum_y \|\hat{h}_{xy}\| \le h_0 \text{ for } \forall x.$$

DATE

§ Absence of SSB under ext. field.

add a symmetry breaking field

hermitian

$$\hat{B}_L(t) = \sum_{x\in\Lambda_L} \hat{b}_x(t) \quad \hat{b}_x(t) \text{ acts only on } \mathcal{H}_x$$

$$\|\hat{b}_x(t)\| \leq b_0$$

a natural choice $\hat{B}_L(t) = \hat{A}_L \cos(\omega t)$

$$\hat{H}_L^{\varepsilon}(t) = \hat{H}_L - \varepsilon \hat{B}_L(t)$$

encourges oscilation.

Φ_{GS}^{ε} : the g.s. of $\hat{H}_L^{\varepsilon}(0)$

Theorem

$$\lim_{\varepsilon\downarrow 0} \lim_{L\uparrow\infty} \frac{1}{L^d} \langle \Phi_{GS}^{\varepsilon}, (\hat{U}_L^{\varepsilon}(t))^{\dagger} \hat{A}\, \hat{U}_L^{\varepsilon}(t)\, \Phi_{GS}^{\varepsilon} \rangle$$

is independent of t

NO SSB (at least for this class of SB field)

proof uses Lieb-Robinson bound.