Part 2

"Quantum spin liquid" in the ground states of low dimensional quantum spin systems

Haldane gap / for $S=1$ AF chains

and the VBS state

Quantum Heisenberg AF $\quad [\hat{H}, \hat{O}^{(\alpha)}] \neq 0 \rightarrow$ "quantum fluctuation" in the g.s.

two spins $\hat{H} = \hat{S}_x \cdot \hat{S}_y \rightarrow$ the g.s. is a spin-singlet

$\rightarrow$ NO ORDER!

$d \geq 2$ the g.s. develops long-rande Néel order.

"quantum fluctuation" is smaller

"quantum fluctuation" is larger

$d = 1$ no long-range Néel order in the g.s.

$\downarrow$

?

<Haldane conjecture and related results>

$d=1$ (almost throughout the present part)

§ Haldane conjecture

Heisenberg AF chain $\hat{H} = \sum_{x=1}^{L} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1}$ $(\hat{\mathbf{S}}_{L+1} = \hat{\mathbf{S}}_1)$

$(S = \frac{1}{2}, 1, \frac{3}{2}, \cdots)$ (L even)

(Marshll-Lieb-Mattis theorem
→ the g.s. is unique for finite L.)

$\boxed{S=\frac{1}{2}}$ Common beliefs based on the Bethe ansatz solution (1931)

i) the g.s. is unique (also for $L \uparrow \infty$) → NO LRO or SSB

ii) no energy gap above the g.s. energy $E_{1st} - E_{GS} = O(\frac{1}{L})$

iii) the g.s. correlation funct. decays by a power law as

$$\langle \Phi_{GS}, \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y \Phi_{GS} \rangle \approx (-1)^{x-y} |x-y|^{-1}$$

Haldan 1983

- non-linear σ-model with a topological term
- sem-classical quantization of solitons

large S limit

$S=\frac{1}{2}, \frac{3}{2}, \cdots$ half-odd-integer spins

i)
ii)
iii)

as in $S=\frac{1}{2}$

massless or critical

$S=1,2,3,\cdots$ integer spins

i)' the g.s. is unique (also for $L\uparrow\infty$) → NO LRO or SSB

ii)' $\exists$ a nonvanishing energy gap above the g.s energy

Haldane gap ($\Delta E \simeq 2Se^{-\pi S}$) $\updownarrow O(1)$

iii)' the g.s. correlation function decays exponentially

$$\langle \Phi_{GS}, \hat{S}_x \cdot \hat{S}_y \Phi_{GS} \rangle \simeq (-1)^{x-y} \exp\left[-\frac{|x-y|}{\xi}\right]$$

massive or disordered

disordered (massive) behavior at $T=0$

strong "quantum fluctuation"

at least in mid 80's

Surprising points of the conjecture

- a drastic difference between the systems with half-odd-integer S and integer S

- it is natural that a one-dim. system with a continuous symmetry has low-energy excitations.

see the next section.

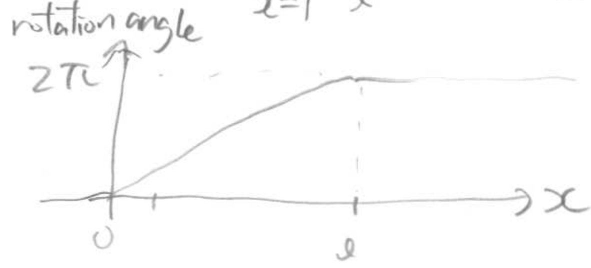
Rem

ii') $\Rightarrow$ iii') was finally proved by Hastings and Koma 2006

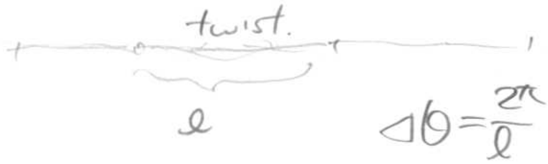
(the beginning of modern applications of the Lieb-Robinson bound)

§ Theorem which rules out "unique g.s. + gap"

twist operator $\hat{U}_\ell = \exp\left[i \sum_{x=1}^{\ell} \frac{2\pi x}{\ell} \hat{S}_x^{(3)} \right]$ $\ell < L$



$\Psi = \hat{U}_\ell \Phi_{GS}$



$$\langle \Psi, \hat{H} \Psi \rangle - E_{GS} = \ell \cdot O((\Delta\theta)^2) = O\left(\frac{1}{\ell}\right)$$

always gapless ?!

One can prove $\langle \Phi_{GS}, \Psi \rangle = 0$ only for $S = \frac{1}{2}, \frac{3}{2}, \cdots$

<u>Theorem</u> (Lieb-Schultz-Mattis 1961, Affleck-Lieb 1986)

For $S = \frac{1}{2}, \frac{3}{2}, \cdots$ "unique g.s. + gap" is impossible.

No information for $S = 1, 2, \cdots$

generalization

(Yamanaka - Oshikawa - Affleck 1997)

§ Semi-classical approach

$$\hat{H} = \underbrace{\sum_{x=1}^{L} \hat{S}^{(3)}_x \hat{S}^{(3)}_{x+1}}_{\text{classical (Ising)} \,\to\, \hat{H}_c} + \underbrace{\frac{1}{2}\sum_{x=1}^{L} \left\{ \hat{S}^+_x \hat{S}^-_{x+1} + \hat{S}^-_x \hat{S}^+_{x+1} \right\}}_{\text{"quantum"} \,\to\, \hat{H}_Q}$$

treat as "perturbation"

$S = \frac{1}{2}$ G.S. of $\hat{H}_c$

↑ ↓ ↑ ↓ ↑ ↓ ↑ ↓

$\hat{S}^+\hat{S}^-$

↑ ↓ ↑ ↑ ↓ ↓ ↑ ↓

$\hat{S}^+\hat{S}^-$

↑ ↓ ↑ ↑ ↓ ↑ ↓ ↓

- <u>pair creation</u> of kinks
- kinks <u>hop</u> by twice the lattice spacing

also <u>pair annihilation</u>.

Note that there are <u>two kinds of kinks</u> → even, odd

different kinds of kinks never pair-annihilate

↑ ↓ ↑ ↑ ↓ ↑ ↑ ↓ ↑

↑ ↓ ↑ ↑ ↑ ↓ ↑ ↓ ↑

no way!!

$\underline{S=1}$ GS of $\hat{H}_c$ $+ - + - + - + -$

$\hat{S}^- \hat{S}^+$

$+ - 0\ 0 + - + -$ pair creation of kinks (0's)

$S^+ S^-$

$+ - 0 + 0 - + -$ kinks hop by a single lattice spacing

$S^- S^+$

$+ - 0 + - 0 + -$

- only one kind of kinks, pairly created and annihilated.

⇑

$\boxed{\text{essential difference from the } S=\tfrac{1}{2} \text{ case}}$

- this construction generates only special states like

$+ 0 - + - 0\,0 + 0 - + 0 - 0 + \cdots$

+ and − alternate with arbitrary number of 0's in between them. → (hidden AF order)

$\tilde{\mathcal{H}}$: restricted Hilbert space generated by these basis states

Theorem (Tasaki '86 unpublished)

The Heisenberg AF on $\tilde{\mathcal{H}}$ has a unique g.s. with a gap and exponentially decaying correlation function

<AKLT model and the VBS picture>

§ AKLT model for $S=1$

$S=1$ (AF) chain with

$$\hat{H}_{AKLT} = \sum_{x=1}^{L} \left\{ \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1} + \frac{1}{3}\left(\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1}\right)^2 \right\}$$

still AF, and $SU(2)$ invariant

<u>Theorem</u> (Affleck-Kennedy-Lieb-Tasaki 1987)

- The g.s. is unique (for finite and infinite L)
- $\exists$ a nonvanishing energy gap (uniform in L)
- $\langle \Phi_{GS}, \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y \Phi_{GS} \rangle = (-1)^{|x-y|} 4 \cdot 3^{-|x-y|}$ $(|x-y| \geq 2)$

strong support to the Haldane conjecture
→ <u>BUT NOT A PROOF !!</u>

a stability theorem (very difficult but important)

<u>Theorem</u> (Yarotsky 2006)

$\hat{V}$: any short ranged translation invariant interaction

$\hat{H} = \hat{H}_{AKLT} + \varepsilon \hat{V}$ For suff. small ε,

the g.s. is unique, $\exists$ a gap, exp. decay.

§ VBS (valence-bond-solid) state

exact g.s. of the AKLT model

$$\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1} + \frac{1}{3}(\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1})^2 = 2\hat{P}_2(\hat{\mathbf{S}}_x + \hat{\mathbf{S}}_{x+1}) - \frac{2}{3}$$

the e.v. of $(\hat{\mathbf{S}}_x + \hat{\mathbf{S}}_{x+1})^2 \to S'(S'+1)$ with $S' = 0, 1, 2$.

$\hat{P}_2$: the proj. onto the space with $S' = 2$

$\hat{H}_{AKLT}$ is essentially the same as

$$\hat{H}'_{AKLT} = \sum_{x=1}^{L-1} \hat{P}_2(\hat{\mathbf{S}}_x + \hat{\mathbf{S}}_{x+1})$$

We shall construct Φ_{VBS} s.t. $\hat{P}_2(\hat{\mathbf{S}}_x + \hat{\mathbf{S}}_{x+1})\Phi_{VBS} = 0$ for $\forall x$.

Then it is a g.s. of $\hat{H}'_{AKLT}$ (and $\hat{H}_{AKLT}$)

construction of the VBS state

- two $S=\frac{1}{2}$'s. $\overset{L}{\uparrow}$ $\overset{R}{\downarrow}$ $\quad \psi_L^{\sigma} \otimes \psi_R^{\sigma'} \quad \sigma, \sigma' = \uparrow, \downarrow$

$\hat{\mathcal{S}}$ → symmetrization

$$\hat{\mathcal{S}}(\psi_L^{\sigma} \otimes \psi_R^{\sigma'}) = \underbrace{\frac{1}{2}\{\psi_L^{\sigma} \otimes \psi_R^{\sigma'} + \psi_L^{\sigma'} \otimes \psi_R^{\sigma}\}}_{\text{total spin } 1.}$$

→ projection op. onto the supspace with $S_{tot}=1$.

- duplicated chain. with sites $(x, L), (x, R)$ $\quad x=1, \cdots, L$

$x-1$ x $x+1$

L R   L R   L R   ....

put $S=\frac{1}{2}$'s. on each site

$$\Phi_{\text{pre-VBS}} := \bigotimes_{x=1}^{L} \underbrace{\frac{1}{\sqrt{2}}\{\psi_{x,R}^{\uparrow} \otimes \psi_{x+1,L}^{\downarrow} - \psi_{x,R}^{\downarrow} \otimes \psi_{x+1,L}^{\uparrow}\}}$$

↓

singlet pair = valence-bond

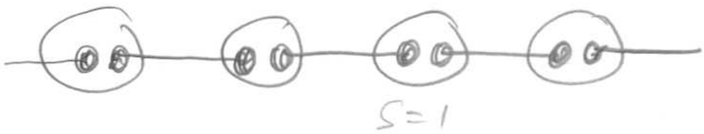
↓

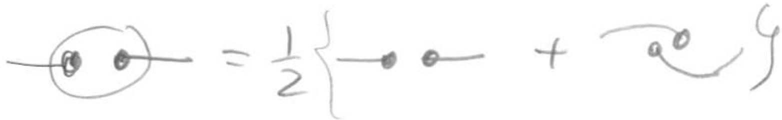
a state for $2L$ spin $\frac{1}{2}$'s.

$$\Phi_{VBS} := \left(\bigotimes_x \hat{\mathcal{S}}_x\right) \Phi_{pre\text{-}VBS}$$

valence-bond solid state

a state for the S=1 chain.





BUT note that

$$\hat{P}_2(\hat{\mathbf{S}}_x+\hat{\mathbf{S}}_{x+1})\,\Phi_{VBS} = \hat{P}_2(\hat{\mathbf{S}}_x+\hat{\mathbf{S}}_{x+1})\left(\bigotimes_x \hat{\mathcal{S}}_x\right)\Phi_{pre\text{-}VBS}$$

$$= \left(\bigotimes_x \hat{\mathcal{S}}_x\right) \underbrace{\hat{P}_2(\hat{\mathbf{S}}_{x,L}+\hat{\mathbf{S}}_{x,R}+\hat{\mathbf{S}}_{x+1,L}+\hat{\mathbf{S}}_{x+1,R})\,\Phi_{pre\text{-}VBS}}_{=\,0}$$

Φ_{VBS} is an exact g.s. of $\hat{H}_{AKLT}$

The theorem is proved based on the exact g.s. and the special properties of the model

gap : a simpler proof Knabe 88

(general theory Fannes, Nachtergaele, Werner 92)

SKIP

§ Proof of the eistence of a gap (Knabe, 1988)

When $E_{GS} = 0$, $\text{gap} \geq \varepsilon \iff \hat{H}^2 \geq \varepsilon \hat{H}$

show this

Write $\hat{P}_x = \hat{P}_2(\hat{S}_x + \hat{S}_{x+1})$

note $\hat{P}_x \hat{P}_y \geq 0$ unless $|x-y| = 1$

$$\hat{H} = \sum_{x=1}^{L} \hat{P}_x \quad (\text{pbc})$$

fix $n \geq 2$, and let $\hat{h}_x = \sum_{y=x}^{x+n-1} \hat{P}_x$

then

$$\sum_{x=1}^{L} (\hat{h}_x)^2 = n \sum_{x=1}^{L} \hat{P}_x + (n-1) \sum_{|x-y|=1} \hat{P}_x \hat{P}_y + (n-2) \sum_{|x-y|=2} \hat{P}_x \hat{P}_y$$

$$+ \cdots + \sum_{|x-y|=n-1} \hat{P}_x \hat{P}_y$$

$$\leq n \sum_{x=1}^{L} \hat{P}_x + (n-1) \sum_{x \neq y} \hat{P}_x \hat{P}_y$$

$$= \underbrace{\sum_{x=1}^{L} \hat{P}_x}_{= \hat{H}} + (n-1) \hat{H}^2.$$

$$\therefore \hat{H}^2 \geq -\frac{1}{n-1} \hat{H} + \frac{1}{n-1} \sum_{x=1}^{L} (\hat{h}_x)^2$$

use $(\hat{h}_n)^2 \geq \varepsilon_n \hat{h}_n$ ($\varepsilon_n > 0$: the gap of $\hat{h}_x$)

SKIP

$$\hat{H}^2 \geq -\frac{1}{n-1}\hat{H} + \frac{1}{n-1}\varepsilon_n \overbrace{\sum_{x=1}^{L}\hat{h}_x}^{=n\hat{H}}$$

$$= \frac{n}{n-1}\left(\varepsilon_n - \frac{1}{n}\right)\hat{H}$$

So $\hat{H}$ has a nonvanishing gap (indep. of L)

if $\varepsilon_n - \frac{1}{n} > 0$ for some n

check numerically.

VBS-1**

Extend the method of S. Knabe J. Stat. Phys. 52, 627-638 (1988) to prove the existence of a nonvanishing gap of the Majumdar-Ghosh model

$S = 1/2$ periodic chain with L even

$$H_{MG} = \sum_{x=1}^{L}\left\{\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1} + \frac{1}{2}\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+2}\right\}$$

~~hint~~ See also Section 5 of AKLT 88.

§VBS state in the standard basis — hidden AF order

$$\circ\!\!-\!\!\!-\!\!\circ = (\uparrow - \downarrow) - (\downarrow - \uparrow)$$

Φ_{VBS} is a sum of many basis states

$$-\downarrow\uparrow - \downarrow\downarrow - \uparrow\downarrow - \uparrow\uparrow - \downarrow\uparrow - \downarrow\uparrow - \downarrow\downarrow - \uparrow\uparrow -$$

$$0 \quad - \quad 0 \quad + \quad 0 \quad 0 \quad - \quad + \quad \cdots$$

+ and − alternate with arbitrary numbers of 0's in between them!

$$\Phi_{VBS} = \sum_{\sigma}{}' (-1)^{z(\sigma)}\, 2^{n(\sigma)/2}\, \Psi^{\sigma}$$

satisfies the constraint

$z(\sigma)$ the number of 0's on odd sites

$n(\sigma)$ the number of $\pm$'s.

- "quantum spin liquid" with hidden AF order

- one gets exactly the same expansion whatever "quantization axis" is taken.

standard AF order → appears in a specific direction

hidden AF order → appears in any directions!

$$\left(\text{rem.}\ \mathcal{S}(\uparrow\uparrow) = \psi^{+} \quad \mathcal{S}(\downarrow\downarrow) = \psi^{-} \atop \mathcal{S}(\uparrow\downarrow) = \frac{1}{2}(\uparrow\downarrow + \downarrow\uparrow) = \frac{1}{\sqrt{2}}\psi^{0} \right)$$

the complicated coefficient with ~~a~~ the ~~long~~ range constraint can be expressed using matrix products.

Fannes, Nachtergaele, Werner 89, 92

Klümper, Schadschneider, Zittarz 91

$$\Phi_{VBS} = \sum_{\sigma}{}' \, \mathrm{Tr}[A_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_n}] \, \Psi^{\sigma} \quad — (*)$$

no constraints $A_+ = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}$, $A_- = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$, $A_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Finitely correlate states
Matrix product states (MPS)

very general but still very special!

VBS-2

Confirm (*) (starting from the def. of VBS)

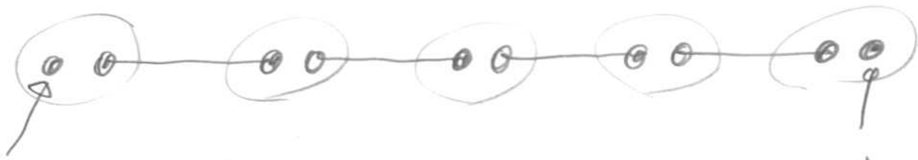
Find a similar expression for the $S=\frac{3}{2}$ state

§ VBS states open chains — edge states

AKLT model on periodic chain, infinite chain

→ the g.s. is unique

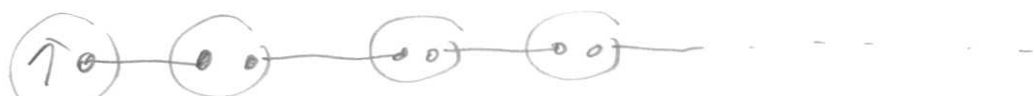
on an open chain



can be anything ~~to~~ ↑or↓ ↑or↓

there are four ground states

~~Semi-infinite~~ long chain with extra ↑

$\Phi'_{VBS} =$ 

the edge spin is not completely localized

$$\langle \Phi'_{VBS}, \hat{S}^{(1)}_x \Phi'_{VBS} \rangle = \langle \Phi'_{VBS}, \hat{S}^{(2)}_x \Phi'_{VBS} \rangle = 0$$

$$\langle \Phi'_{VBS}, \hat{S}^{(3)}_x \Phi'_{VBS} \rangle = -2(-3)^{-x}$$

$$\sum_{x=1}^{\infty} \langle \qquad \rangle = \frac{1}{2}$$

$\Lambda_L = \{-\frac{L}{2}+1, -\frac{L}{2}+2, \cdots, \frac{L}{2}\}$

the four g.s. converges to a single inf. vol gr. st.
as $L \uparrow \infty$

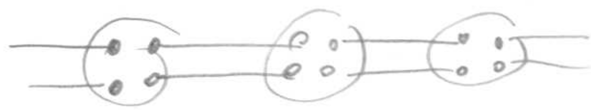
recall that

	finite L	$L \uparrow \infty$
Heisenberg AF $d \geq 2$	unique g.s.	infinitely many g.s.
AKLT open chain	four g.s.	unique g.s

§ VBS picture

Can we form VB* states for other S ?

$S=2$ =  four $S=\frac{1}{2}$'s



$S=\frac{3}{2}$ =  three $S=\frac{1}{2}$'s

translation inv. is broken.

We can construct translation invariant VBS only for integer S.

skip

BUT under magnetic field one may have a g.s. states like

for $S=\frac{3}{2}$ chain

VBS like state

here $S^{(3)}_{Tot}=\frac{L}{2}$

YOA filling factor is $\nu=\frac{1}{2}+\frac{3}{2}=2$

integer!

⟨Haldane phase⟩

§ Haldane conj. for the S=1 Heisenberg AF chain

$$\hat{H} = \sum_{x=1}^{L} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1}$$

numerical results → gap is also observed experimentally!

- $\exists$ a gap $\simeq 0.41$ above the unique g.s.
- correlation in g.s. decays exponentially

BUT STILL NO PROOF → [AKLT is at the "center" of the "Haldane phase", and the Heisenberg AF happens to belong to that phase ? ? ?]

§. The model with anisotropy

S=1 chain (pbc)

$$\hat{H}_{aniso} = \sum_{x=1}^{L} \left\{ \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1} + \underbrace{D(\hat{S}_x^{(3)})^2}_{\text{anisotropy}} \right\} \qquad D \geq 0$$

note that

$$\hat{H}_0 = \sum_{x=1}^{L} D(\hat{S}_x^{(3)})^2 \quad \text{is trivial}$$

G.S. $\Phi_0 = \bigotimes_{x=1}^{L} \psi_x^0$ 0 0 0 0 0 0 0

$E_0 = 0$

1st excited   0 0 0 + 0 0 0   or   0 0 0 − 0 0 0

$E_{1st} = E_0 + D$ ← energy gap

if $D \gg 1$

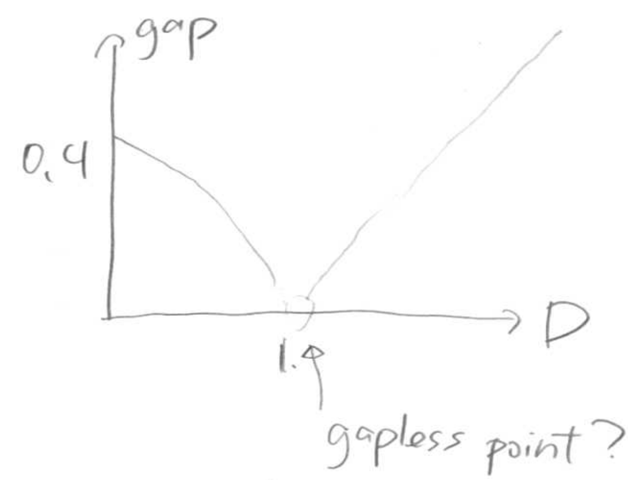
- The g.s. is unique and is close to Φ_0
- $\exists$ a gap $\simeq D$
- The g.s. correlation decays exponentially

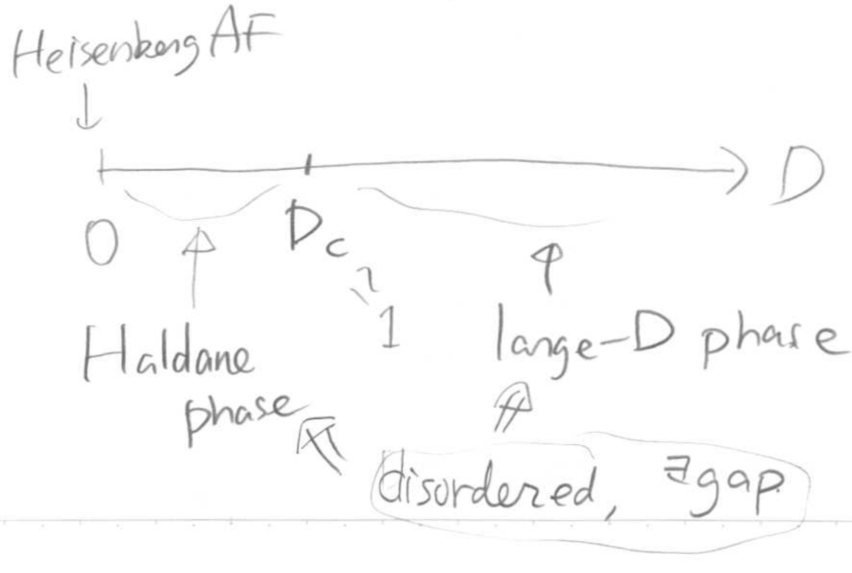
all rigorous and trivial

(cluster expansion)

Is the Haldane gap smoothly connected to this trivial gap?

numerical results





§ Peculiar features of the Haldane phase

Hidden AF order

The g.s. of $\hat{H}_{\text{aniso}}$ $\quad \Phi_{GS} = \sum_{\sigma} c_{\sigma} \tilde{\Psi}^{\sigma}$

$$c_{\sigma} > 0 \text{ for } \forall \sigma \quad \text{(Marshall-Lieb-Mattis)}$$

(different from the VBS state)

BUT in the Haldane phase. most states (with considerable weight) look like

$$+\ 0\ -\ +\ -\ 0\ 0\ \underbrace{-\ 0\ +}_{\text{defect}}\ 0\ +\ 0\ -\ +\ -$$

the long-range hidden AF order still presents

$$\underset{\text{even}}{+}\ -\ +\ -\ \underbrace{-\ +}\ +\ -\ \underset{\text{even}}{+}\ -\ +$$

den Nijs - Rommelse string order parameter 1989

$$O^{(\alpha)}_{\text{string}} := -\lim_{|x-y|\uparrow\infty} \lim_{L\uparrow\infty} \left\langle \Phi_{GS}, \hat{S}^{(\alpha)}_x \underbrace{\exp\left[i\pi \sum_{z=x+1}^{y-1} \hat{S}^{(\alpha)}_z\right]}_{= (-1)^{\sum \hat{S}^{(\alpha)}_z}} \hat{S}^{(\alpha)}_y \Phi_{GS} \right\rangle$$

$\alpha = 1, 2, 3$

for the VBS state $O^{(\alpha)}_{string} = \frac{4}{9}$ $\alpha = 1, 2, 3.$

heuristic arguments
+ numerical res. for $\hat{H}_{aniso}$

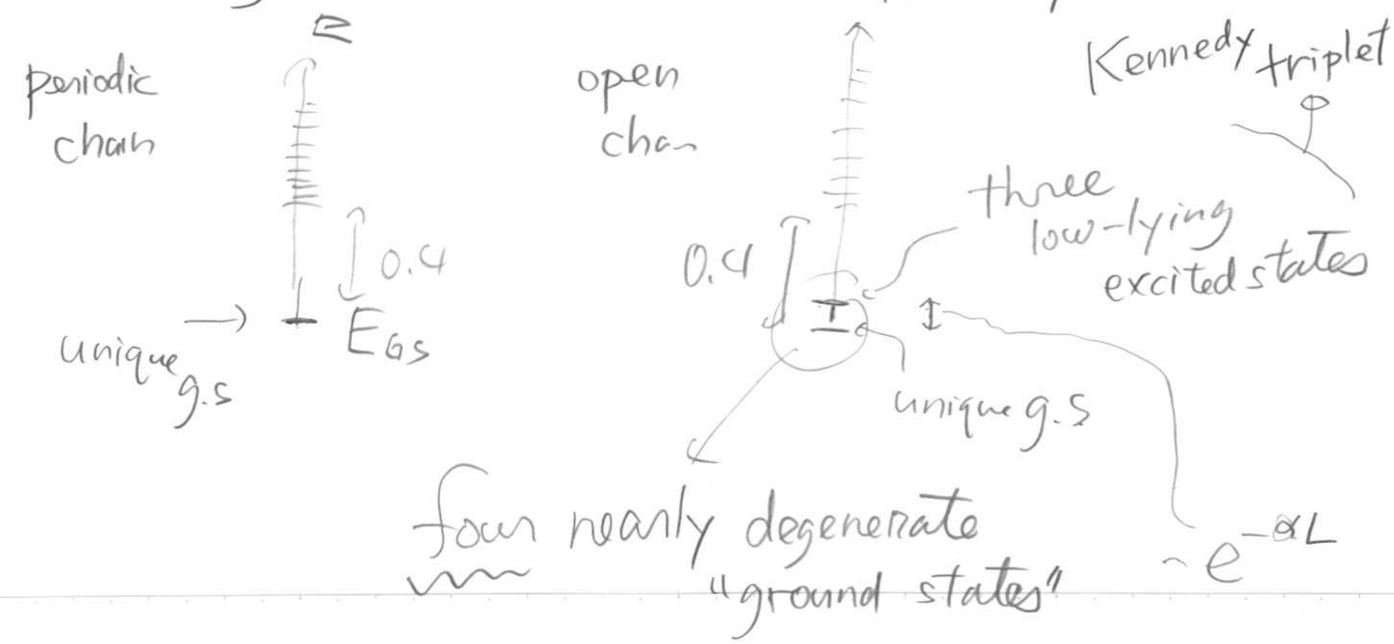
$$\begin{cases} \text{Haldane phase} & O^{(1)}_{string} = O^{(2)}_{string} > 0, \quad O^{(3)}_{string} > 0 \\ \text{large-}D\text{ phase} & O^{(1)}_{string} = O^{(2)}_{string} = O^{(3)}_{string} = 0 \end{cases}$$

> The hidden AF order (measured by the string order par.) characterizes the Haldane phase.

Near four-fold degeneracy and the edge states

- AKLT model on { a periodic chain → unique g.s. + a gap
  { an open chain → four g.s. + a gap

- Heisenberg AF (numerical) Kennedy 1990



hidden AF order $\Rightarrow$ near four-fold degeneracy

1) Harsch-von den Liden Theorem

$$\hat{O}^{(\alpha)}_{\text{string}} := \sum_{x=1}^{L} \hat{S}_x^{(\alpha)} \exp\left[i\pi \sum_{y=1}^{x-1} \hat{S}_y^{(\alpha)}\right]$$

if $O^{(\alpha)}_{\text{string}} \neq 0$

Then $\langle \Phi_{GS}, (\hat{O}^{(\alpha)}_{\text{string}})^2 \Phi_{GS} \rangle \geq \alpha . L^2$ $\alpha > 0$

Thus $\dfrac{\hat{O}^{(\alpha)}_{\text{string}} \Phi_{GS}}{\| \hat{O}^{(\alpha)}_{\text{string}} \Phi_{GS} \|}$ is a low-lying state

$\alpha = 1, 2, 3$

they are orthogonal

2) 0, +, − configuration

config. with complete hidden AF order

$|0 + 0 - + \cdots - + 0 0 -|$

$|0 0 - + \cdots +|$

$|- \cdots -|$

$|+ \cdots +|$

four kinds

edges states

Thus "Haldane phase" is a distinct phase

0 Haldane $D_c \simeq 1$ large-D → D

hidden AF order | no order

near-four-fold degeneracy in open chain (edge states) | unique GS with a gap in open chain

quite exotic!

observed experimentally!

§ Non-local unitary transformation and hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking

(Kennedy-Tasaki 92)

open chain

$$\hat{H} = \sum_{x=1}^{L-1} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1} + D \sum_{x=1}^{L} \left(\hat{S}_x^{(3)}\right)^2$$

basis state $\Psi^{\sigma} = \bigotimes_{x=1}^{L} \psi_x^{\sigma_x}$ with $\sigma = (\sigma_x)_{x=1,\dots,L}$
$\sigma_x = 0, \pm 1$.

For σ, define $\sigma' = (\tilde{\sigma}_x)_{x=1,\dots,L}$ by

$$\sigma'_x = \underbrace{(-1)^{\sum_{y=1}^{x-1} \sigma_y}} \sigma_x$$

σ 0 0 + 0 − + 0 + − 0 − + hidden AF order

σ' 0 0 + 0 + + 0 − − 0 + + ferro order

local defect

Define unitary op. $\hat{U}$ by

$$\hat{U} \Psi^{\sigma} = (-1)^{N(\sigma)} \Psi^{\sigma'}$$

$N(\sigma)$: the number of odd x with $\sigma_x = 0$.

(Oshikawa's form
$$\hat{U} = \prod_{x<y} \exp[i\pi \hat{S}_x^{(3)} \hat{S}_y^{(1)}]$$
)

$$\left(\begin{array}{l} \hat{H}\,\Phi_{GS} = E\,\Phi_{GS} \\ \hat{H}'\,\Phi'_{GS} = E\,\Phi'_{GS} \qquad \Phi'_{GS} = \hat{U}\,\Phi_{GS} \end{array} \right)$$

Then

$$\hat{H}' = \hat{U}\hat{H}\hat{U}^{\dagger}$$

$$= \sum_{x=1}^{L-1} \left\{ -\underbrace{\hat{S}_x^{(1)}\hat{S}_{x+1}^{(1)}} + \hat{S}_x^{(2)} e^{i\pi(\hat{S}_x^{(3)}+\hat{S}_{x+1}^{(3)})} \hat{S}_{x+1}^{(2)} - \underbrace{\hat{S}_x^{(3)}\hat{S}_{x+1}^{(3)}} \right\}$$

$$+ D \sum_{x=1}^{L} \left(\hat{S}_x^{(3)}\right)^2$$

is

- mainly ferromagnetic (especially in the 1st and the 3rd directions)

- has a discrete symmetry

  invariant under the π-rotation around the $\underbrace{1, 2, \text{or } 3}$ axis.

  $\uparrow$ not independent.

  $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \text{ symmetry}}$

the long-range order parameters of the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry breaking

$$O_{\text{ferro}}^{(\alpha)} = \lim_{|x-y|\uparrow\infty} \langle \Phi'_{GS}, \hat{S}_x^{(\alpha)}\hat{S}_y^{(\alpha)}\,\Phi'_{GS} \rangle \qquad (\alpha = 1, 3)$$

$$\Phi'_{GS} = \hat{U}\,\Phi_{GS}$$

then it holds that

$$\boxed{O_{\text{ferro}}^{(\alpha)} = O_{\text{string}}^{(\alpha)} \qquad (\alpha=1,3)}$$

Φ'_{GS} $\qquad\qquad$ Φ_{GS}

The picture of hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking

$\hat{H}'$: ferromagnetic Hamiltonian with discrete $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

- large-D phase $D > D_c$
  no symmetry breaking
  unique g.s. + a gap

- Haldane phase $0 \leq D < D_c$
  $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is fully broken
  - SSB of a discrete symmetry $\rightarrow$ gap
  - ferromagnetic order $\rightarrow$ hidden AF order
  - four g.s. in the infinite chain
    $\hookrightarrow$ four low-lying energy excitations in a finite chain

($\hat{H}$ and $\hat{H}'$ have exactly the same spectra)

$$\boxed{\text{all the exotic properties of the Haldane phase can be understood as a consequence of the } \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ symmetry breaking.}}$$

$\rightarrow$ starting point of other rigorous and non-rigorous theories

⟨Some related issues⟩

§ Stability of the Haldane phase

Does the $\mathbb{Z}_2 \times \mathbb{Z}_2$ picture explain everything?

- edge states of the $S=2$ VBS

$3 \times 3 = 9$ fold degeneracy.

($\mathbb{Z}_2 \times \mathbb{Z}_2$ suggests four) ?

- string order for the general VBS (Oshikawa 92)

$$O^{(\alpha)}_{\text{string}} \begin{cases} > 0 & \text{for } S = 1, 3, 5, \cdots \\ = 0 & \text{for } S = 2, 4, 6, \cdots \end{cases}$$

?

Is it possible to connect the Haldane and the large-D phases smoothly ?

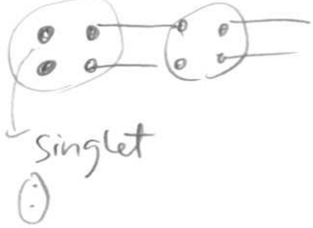
Is there $\hat{H}_\lambda$ such that

- $\hat{H}_\lambda$: Ham on the open chain, depends smoothly on $\lambda \in [0,1]$, has a suitable symmetry (e.g. inv. under $\hat{S}_x \to -\hat{S}_x$ for all x)
- $\hat{H}_0 = \sum_{x=1}^{L} D(\hat{S}_x^{(3)})^2$, $\hat{H}_1 = \hat{H}_{AKLT}$
- $\hat{H}_\lambda$ has a unique g.s. + a gap for $\forall \lambda \in [0,1]$.

?

<u>Yes</u> for $S = 2, 4, 6, \cdots$

edge



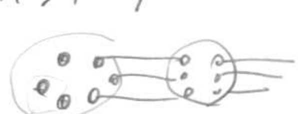
singlet

(Haldane phase is a "symmetry protected topological phase")

but any phase is protected by a symmetry

<u>No</u> for $S = 1, 3, 5, \cdots$

edge

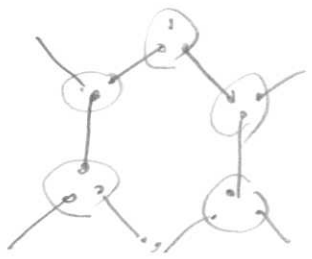


at least $S = \frac{1}{2}$ remains ($\to$ four-fold near degeneracy)

Gu, Wen
Pollmann, Turner, Berg, Oshikawa

§ VBS in two dimensions

$S=\frac{3}{2}$ model on the hexagonal lattice



- g.s. is unique
- g.s. correlation decay exponentially

<u>no proof of gap</u>

hidden order ??
SSB ??? ← no ideas

§ Hamiltonian vs. states

(ground) states are more important than the Ham. → MPS, tensor network, ... New TREND?!
VBS, Laughlin, BCS ...
<u>start from states</u>

<u>BUT</u> <u>philosophically</u>
microscopic physics → low energy effective theories → g.s. or eq. states with interesting physics.
we miss these exciting steps!

<u>practically</u> We miss many important problems (Haldane gap in $S=1$ Heisenberg AF)