アンダーソン局在から トポロジカル絶縁体へ

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Outline

- アンダーソン局在
  – Scaling theory
- トポロジカル絶縁体とトポロジカル超伝導体
  – Examples
  – Symmetry classes
- トポロジカル絶縁体・超伝導体の分類理論

Schnyder, Ryu, AF, and Ludwig, Phys. Rev. B **78**, 195125 (2008)
Ryu, Schnyder, AF, and Ludwig, New J. Phys. **12**, 065010 (2010)

A.P. Schnyder (MPI Stuttgart)
笠 真生 (Univ. Illinois, Urbana-Champaign)
A.W.W. Ludwig (UC Santa Barbara)

Anderson localization

P. W. Anderson

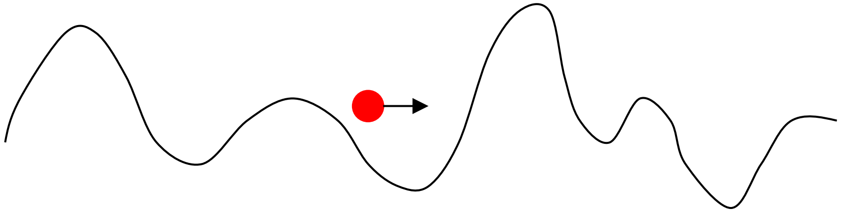
"Absence of diffusion in certain random lattices" Phys. Rev. (1958)

1977年ノーベル賞

$$\left[-\frac{\hbar^2}{2m}\nabla^2+V\left(\vec{r}\right)\right]\psi\left(\vec{r}\right)=E\psi\left(\vec{r}\right)$$

一体問題

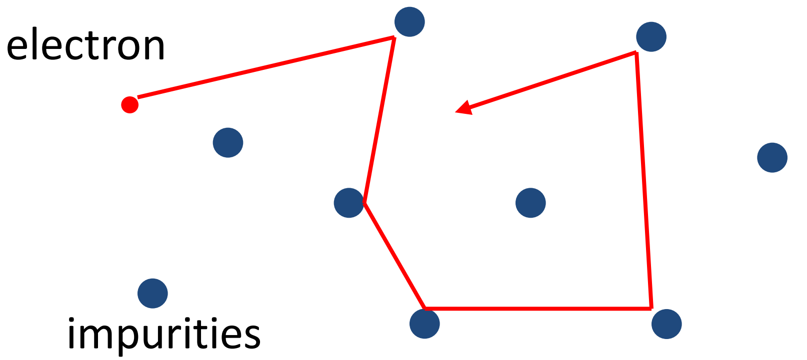
ランダムポテンシャル中を運動する電子



ランダムネスが十分強ければ
ポテンシャルの底近傍に
波動関数は局在する

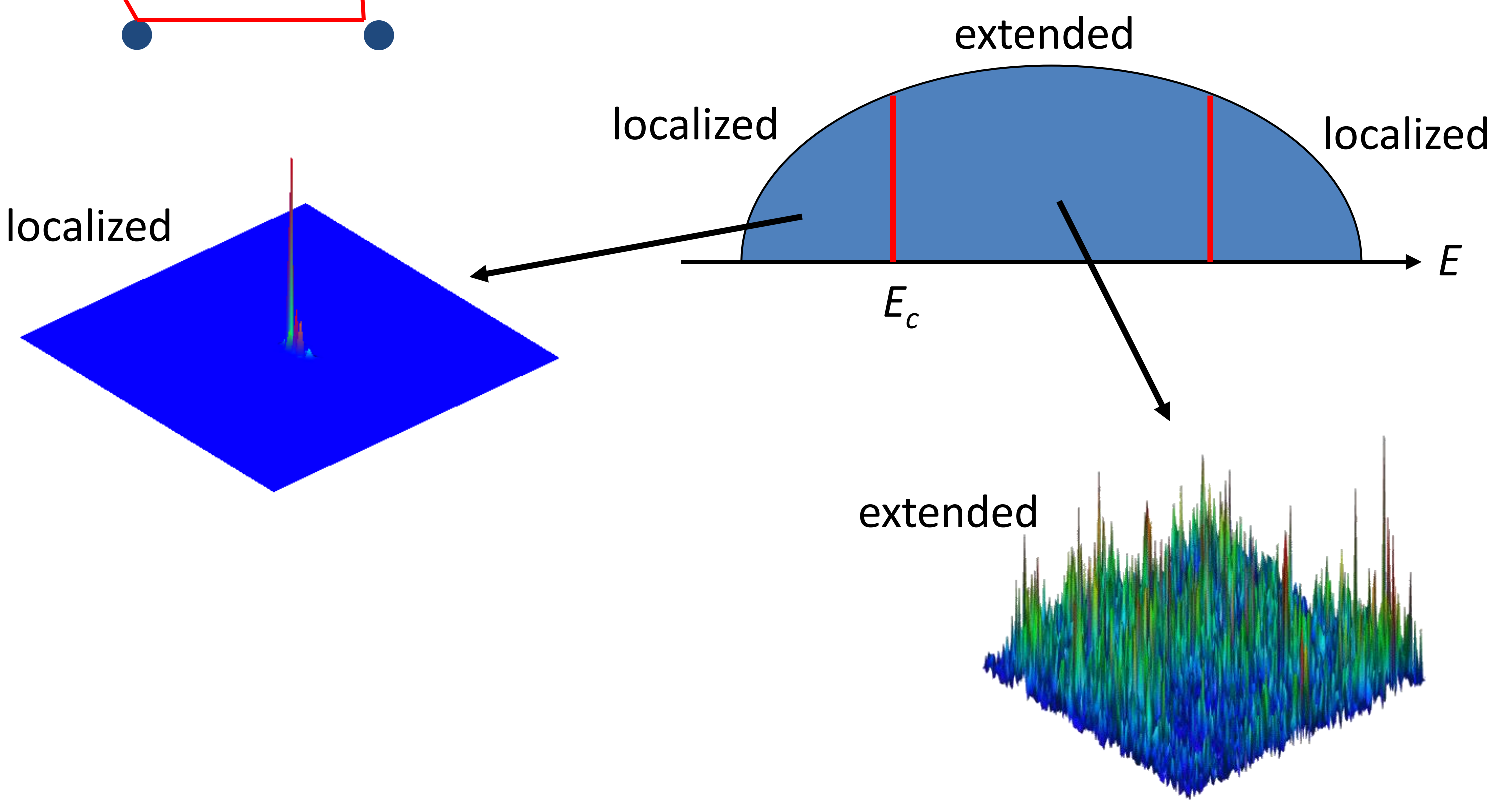
アンダーソン絶縁体

不純物によって散乱されながら運動する電子



Quantum interference of scattering waves

→ Anderson localization of electrons



Scaling theory (Abrahams, Anderson, Licciardello, Ramakrishnan, PRL 1979)

一辺の長さがL の試料のコンダクタンス g

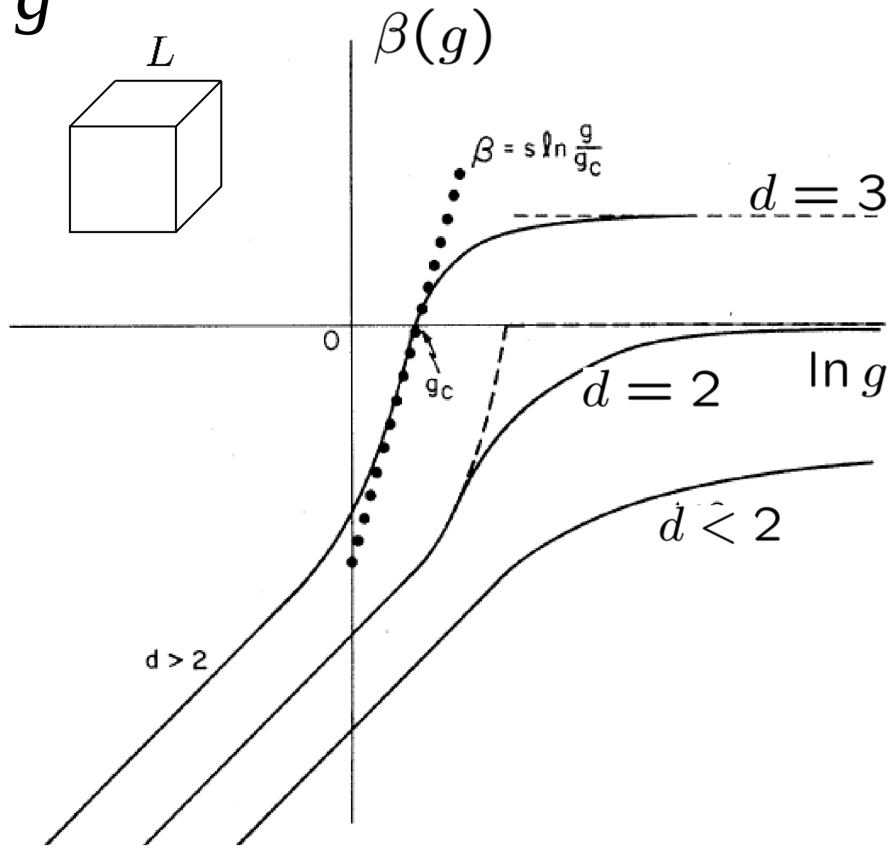
$$g(2L) = F\left[g(L), \cancel{L}\right]$$

Metal: $g \propto \dfrac{\text{area}}{\text{length}} = L^{d-2}$

$$\beta(g) = \frac{d \ln g}{d \ln L} = d - 2 - O(g^{-1})$$

Insulator: $g \propto e^{-L/\xi}$

$$\beta(g) = \ln g + O(g^0)$$



All wave functions are localized below two dimensions!

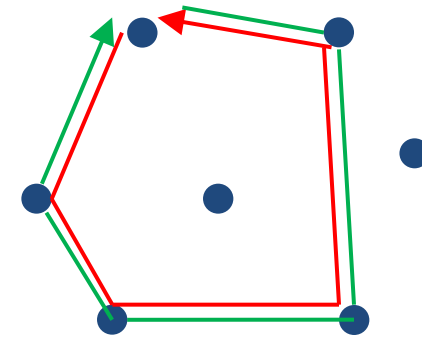
A metal-insulator transition at $g=g_c$ is continuous (d>2).

Prog. Theor. Phys. Vol. 63, No. 2, February 1980, Progress Letters

Spin-Orbit Interaction and Magnetoresistance in the Two Dimensional Random System

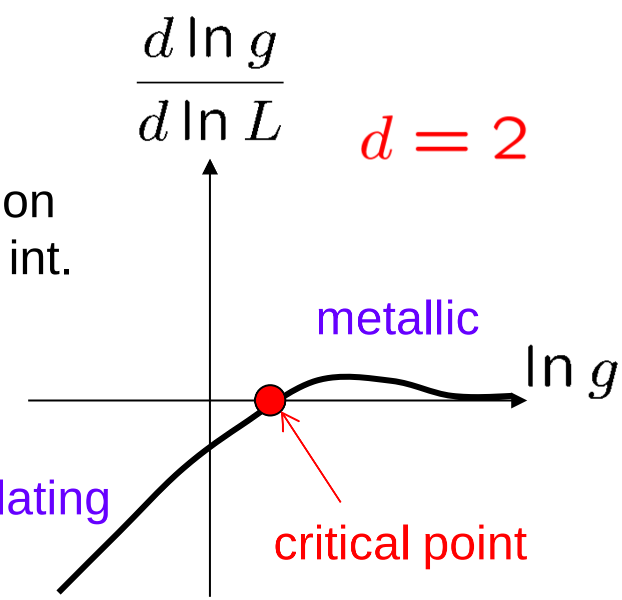
Shinobu HIKAMI, Anatoly I. LARKIN*) and Yosuke NAGAOKA
Research Institute for Fundamental Physics
Kyoto University, Kyoto 606

(Received November 5, 1979)



3 symmetry classes (orthogonal, unitary, symplectic)

Wigner-Dyson RMT ensembles



symplectic class: ○ time-reversal, × spin-rotation w/ spin-orbit int.

anti-localization

$$\frac{d \ln g}{d \ln L} = d - 2 + \frac{c}{g} \qquad c > 0$$

insulating

Metal-insulator transition in 2D

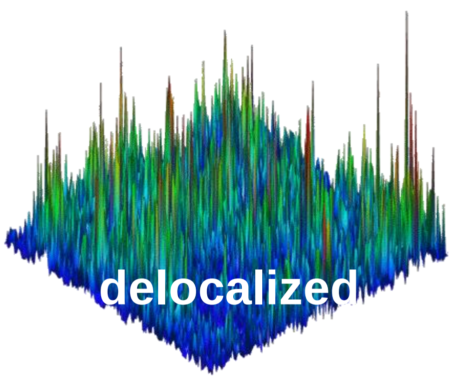
Anderson transition (metal-insulator transition)

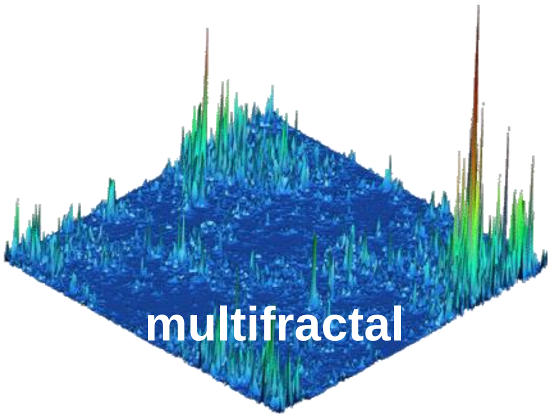
Continuous phase transition induced by disorder

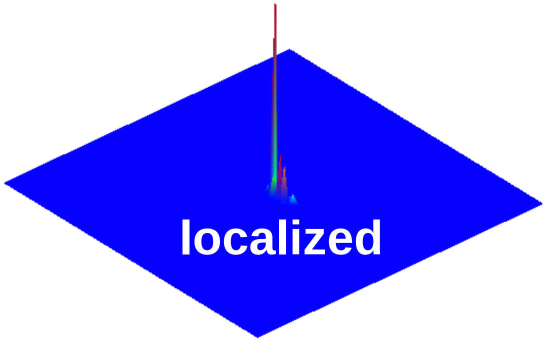
localization length $\xi \to \infty$ $\xi \sim |E - E_c|^{-\nu}$

scale invariance universal critical properties

MIT in the 2d symplectic class $\nu \approx 2.7$ (Asada, Ohtsuki, & Slevin, 2002)







metallic phase critical point insulating phase

Anderson metal-insulator transition is
a continuous quantum phase transition driven by disorder

- Dimensionality *d*

- Symmetry of Hamiltonian

  time-reversal symmetry

  (SU(2) rotation symmetry in spin space)

  Wigner-Dyson ensemble of random matrices

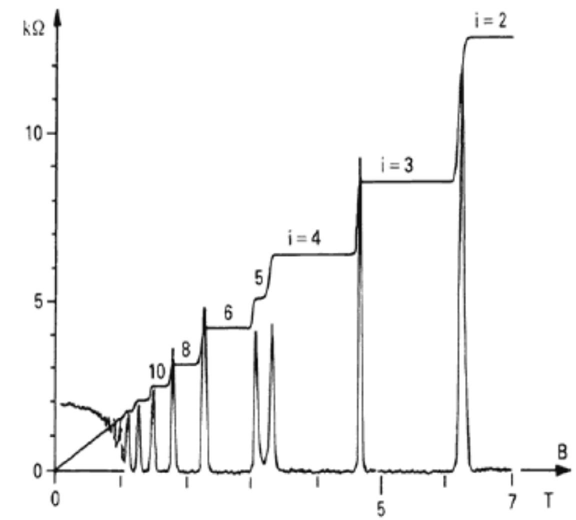
	time reversal symmetry	spin rotation symmetry
orthogonal	yes $T^2 = +1$	yes (S^z保存 → spinlessと同じ)
unitary	no	----
symplectic	yes $T^2 = -1$	no

ユニタリクラス

時間反転で対称でない系：　磁場中の電子系など

スケーリング理論によれば、d=2 では常に局在　$\beta(g)=\frac{d\ln g}{d\ln L}<0$

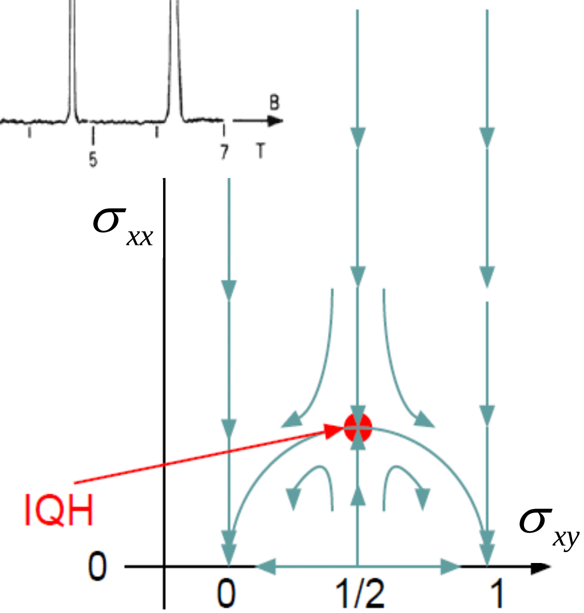
整数量子ホール効果　(von Klitzing 1985)



$$\sigma_{xy}=N\frac{e^2}{h}\qquad \sigma_{xx}=0$$

プラトー間転移(N→ N+1)は臨界点

２パラメータ・スケーリング　(Khmelnitskii, Pruisken)



非線形シグマ模型＋トポロジカル項

数値計算　$\nu\approx 2.3-2.4$

10 random matrix ensembles (symmetric spaces)

Altland & Zirnbauer (1997)

	Cartan label	TRS	PHS	Ch	time evolution operator $\exp(-iHt)$
Wigner-Dyson	A (unitary)	0	0	0	$U(N)$
	AI (orthogonal)	+1	0	0	$U(N)/O(N)$
	AII (symplectic)	−1	0	0	$U(2N)/Sp(2N)$
chiral	AIII (ch. unit.)	0	0	1	$U(N+M)/U(N)\times U(M)$
	BDI (ch. orth.)	+1	+1	1	$O(N+M)/O(N)\times O(M)$
	CII (ch. sympl.)	−1	−1	1	$Sp(N+M)/Sp(N)\times Sp(M)$
superconductor	D (BdG)	0	+1	0	$SO(2N)$
	C (BdG)	0	−1	0	$Sp(2N)$
	DIII (BdG)	−1	+1	1	$SO(2N)/U(N)$
	CI (BdG)	+1	−1	1	$Sp(2N)/U(N)$

- Wigner-Dyson (1951-1963): “three-fold way” complex nuclei
- Verbaarschot & others (1992-1993) chiral phase transition in QCD
- Altland-Zirnbauer (1997): “ten-fold way” mesoscopic superconductors

ランダム平均

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) \qquad \overline{V(\vec{r})} = 0, \quad \overline{V(\vec{r})V(\vec{r}\,')} = u\delta(\vec{r}-\vec{r}\,')$$

物理量 X のランダム平均

レプリカ法

$$\overline{X} = \overline{\frac{\int D\psi\, X e^{-S}}{\int D\psi\, e^{-S}}} = \overline{\frac{1}{Z}\partial_h Z_h} = \lim_{n\to 0}\frac{1}{n}\partial_h \overline{Z^n}$$

n個のreplica
Replica limit $n \to 0$

Z^nに対してランダム平均 ➡ 相互作用する電子系

➡ 相互作用を補助場で表す（HS変換）➡ 電子系を積分 ➡ 補助場に対する有効理論

非線形シグマ模型

Supersymmetry (f: fermion, b: boson)

$$\overline{X} = \int Df Db X\, \overline{e^{-S_f - S_b}} \qquad \int Df e^{-S_f} = \left(\int Db e^{-S_b}\right)^{-1}$$

臨界点の理論: 未解決の難問

- 臨界点は弱結合領域にはない
- 2次元の場合 : 共形対称性
  - 数値的検証 （Obuse, Subramaniam, AF, Gruzberg, Ludwig, 2007, 2010; Zirnbauerら 2013）

    有限サイズスケーリング（Cardy）

    有限幅の系の局在長 ⟺ 2次元臨界波動関数のフラクタル指数
  - $c = 0$ 非ユニタリな共形場理論

    負のスケーリング次元の演算子 （例えば、マルチフラクタル指数）
  - Class C いくつかの指数は厳密に計算できる （= percolation）
- 1次元の場合
  - 転送行列の固有値（Lyapunov指数）に対するFokker-Planck方程式

    DMPK方程式

    (Dorokhov; Mello-Pereyra-Kumar; Beenakker; Brouwer-AF-Mudry-Gruzberg)
  - SUSY nonlinear sigma model: gの低次モーメントの厳密な計算 (AIII, CI, DIII)

    (Lamakraft-Simons-Zirnbauer)

トポロジカル絶縁体
と
トポロジカル超伝導体

以下では、ランダムポテンシャルは（しばらく）考えない。

広い意味での

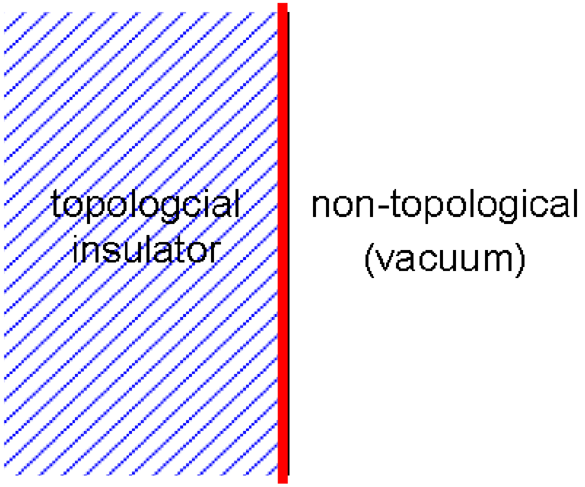
Topological (band) insulators

- band insulators

free fermions (ignore e-e int.)

- characterized by a topological number (Z or Z_2)
- gapless excitations at boundaries

stable



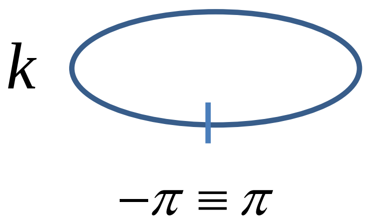
Examples: integer quantum Hall effect,

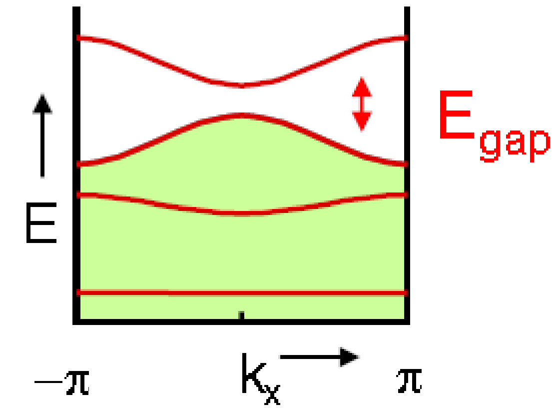
time reversal symmetry ⟹ quantum spin Hall insulator, 3D Z_2 topological insulator, ….

2D 3D

Energy band structure:

map $\vec{k} \mapsto E_n(\vec{k})$, or $\left|u_n(\vec{k})\right\rangle$





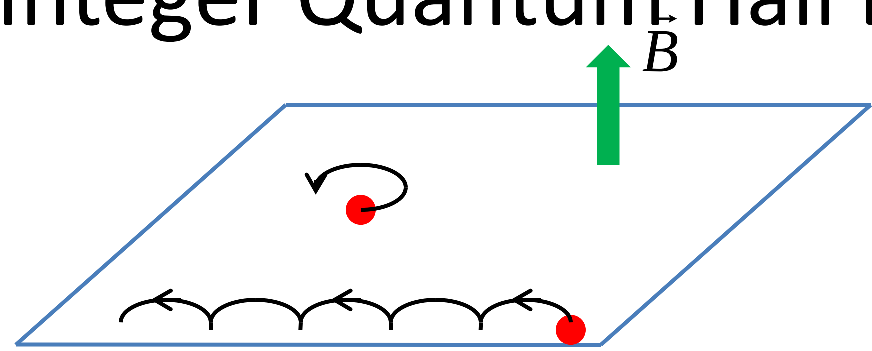
topological numbers (e.g., winding number)

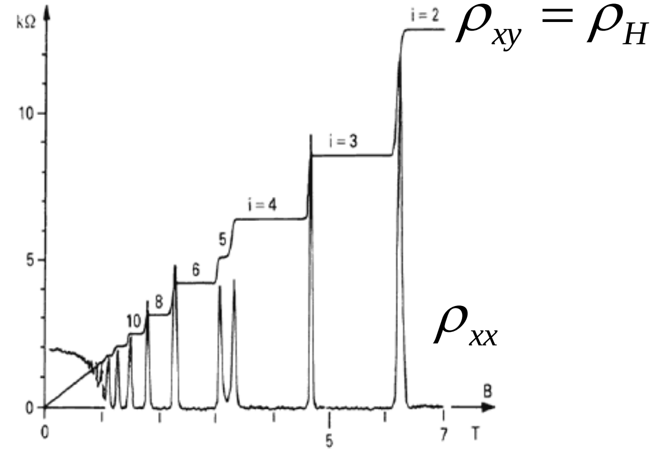
Band structures are topologically equivalent,
if they can be continuously deformed from one to another
without closing the energy gap.

Topological numbers are not changed by continuous deformation.

(discrete number)

Integer Quantum Hall Effect





TKNN number (Thouless-Kohmoto-Nightingale-den Nijs)

$$\sigma_{xy} = -\frac{e^2}{h} C$$

TKNN (1982); Kohmoto (1985)

1st Chern number

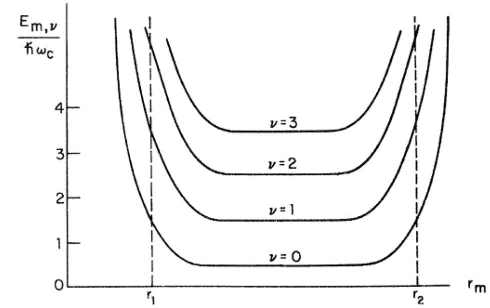
integer valued

$$C = \frac{1}{2\pi i} \int_{\text{filled band}} d^2k\ \vec{\nabla}_k \times \vec{A}(k_x, k_y)$$ = number of edge modes crossing E_F

bulk-edge correspondence

$$\vec{A}(k_x, k_y) = \langle \vec{k} | \vec{\nabla}_k | \vec{k} \rangle$$ Berry connection

$$\vec{\nabla}_k = (\partial_{k_x}, \partial_{k_y})$$

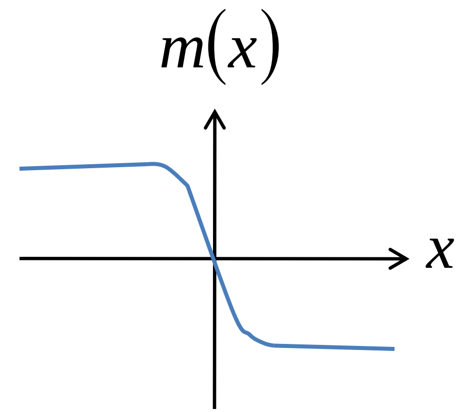


Effective field theory

$$H = -iv\left(\sigma_x \partial_x + \sigma_y \partial_y\right) + m\sigma_z$$

parity anomaly $\Longrightarrow$ $\sigma_{xy} = \frac{1}{2}\mathrm{sgn}(m)$

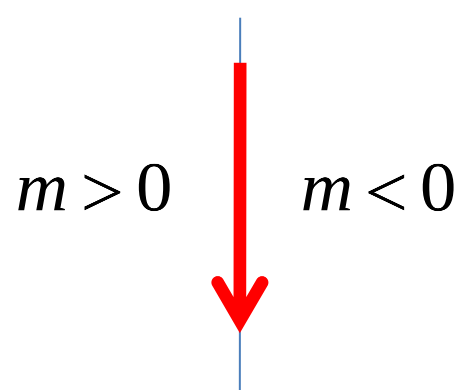
Domain wall fermion



$$H = -iv\left(\sigma_x \partial_x + \sigma_y \partial_y\right) + m(x)\sigma_z$$

$$\psi(x,y) = \exp\left[iky - \frac{1}{v}\sigma_y \int_0^x m(x')dx'\right]\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

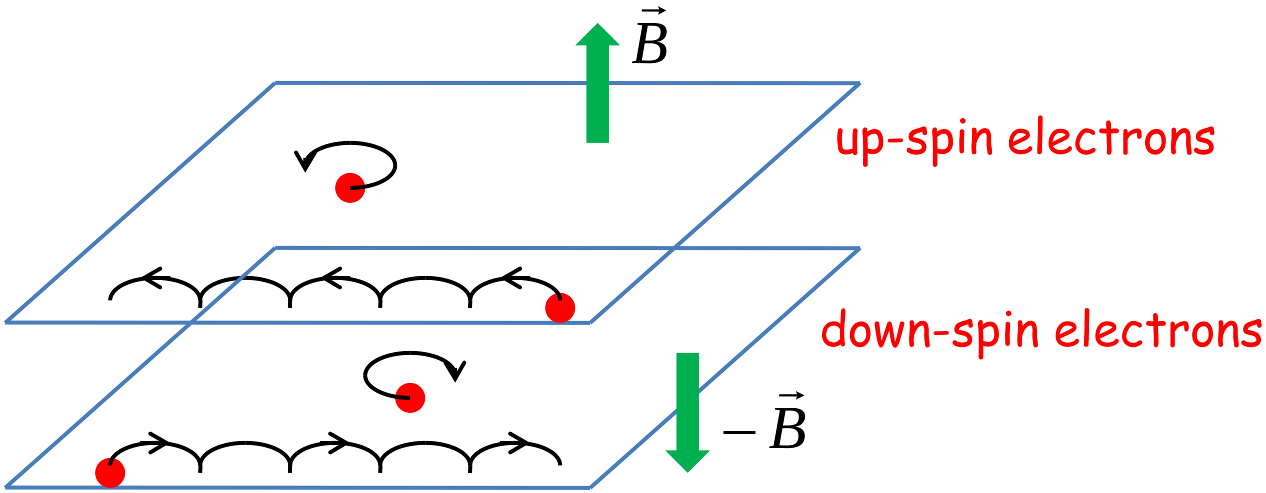
$$E = -vk$$

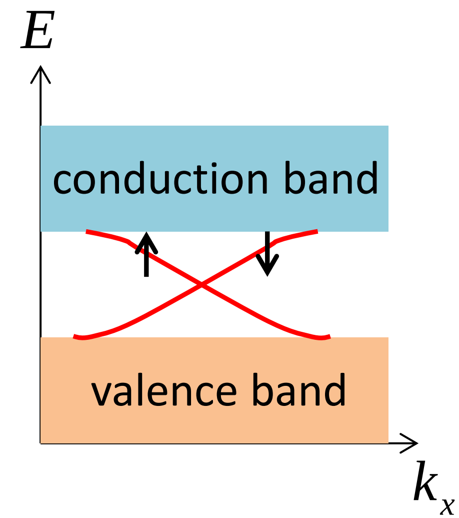


2D Quantum spin Hall effect (2D Z_2 TPI)

Kane & Mele (2005, 2006); Bernevig & Zhang (2006)

- time-reversal invariant band insulator
- spin-orbit interaction
- gapless helical edge mode (Kramers' pair)



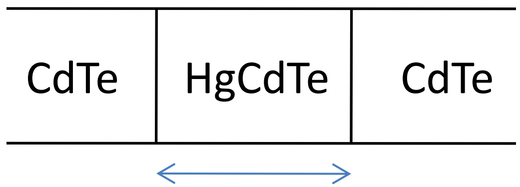


S^z is not conserved in general. Topological index: $Z \Rightarrow Z_2$

Spin flip: Rashba spin-orbit coupling etc.

Experiment

HgTe/(Hg,Cd)Te quantum wells



Konig et al. [Science 318, 766 (2007)]

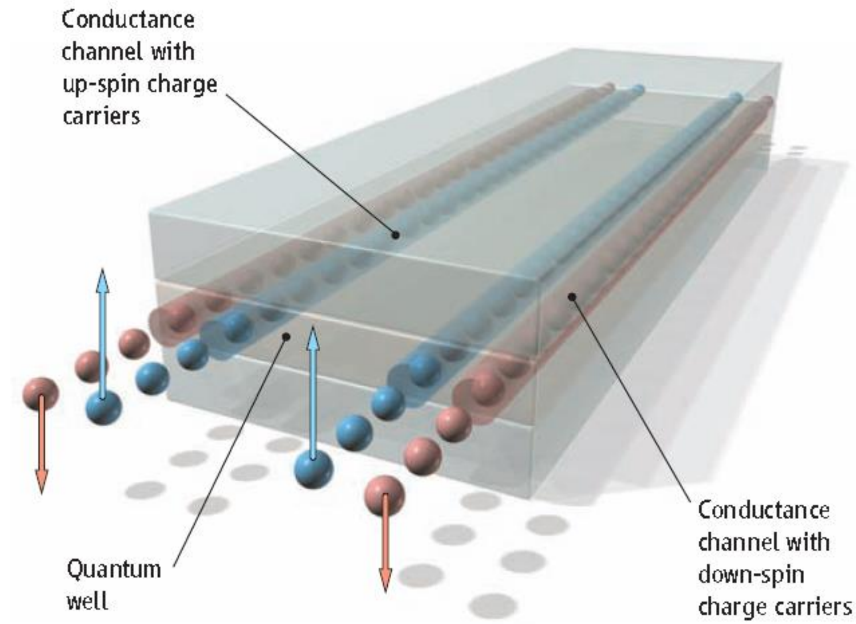
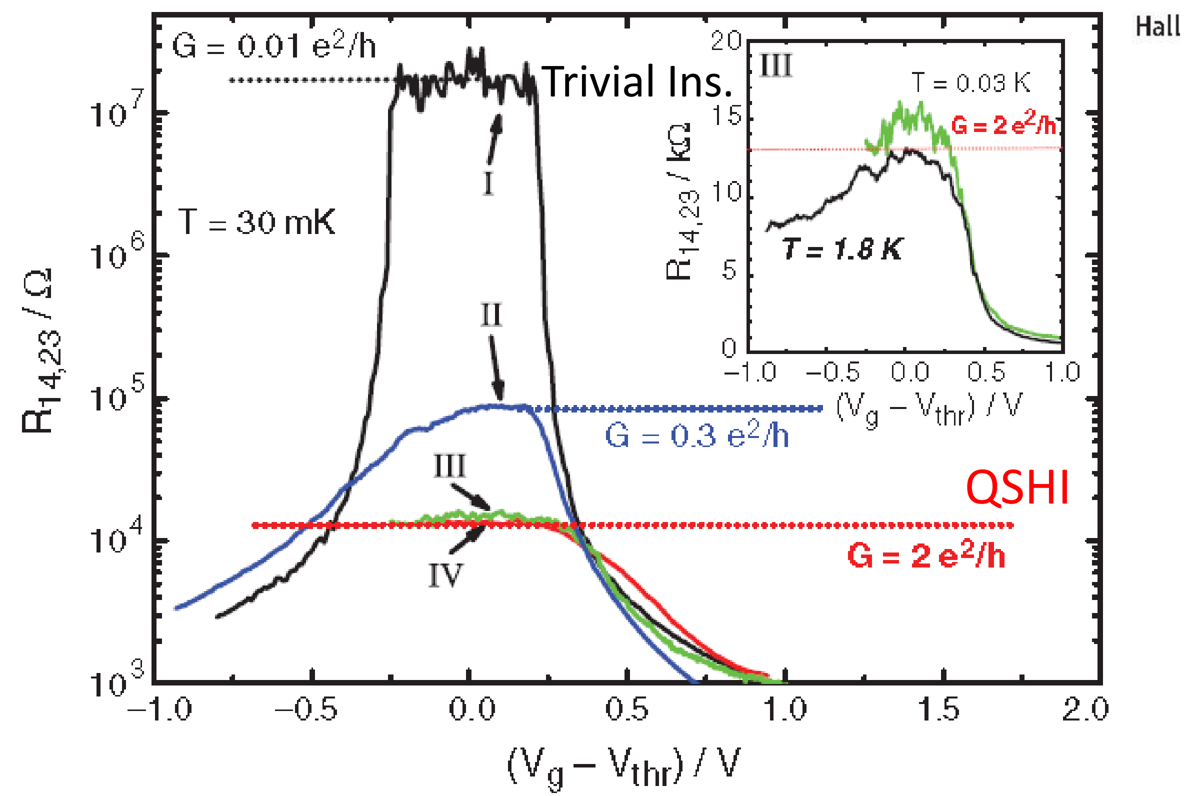
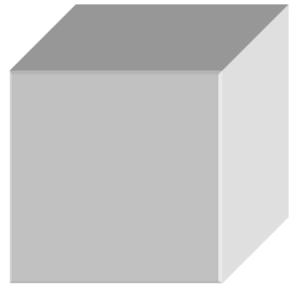


Fig. 4. The longitudinal four-terminal resistance, $R_{14,23}$, of various normal (d = 5.5 nm) (I) and inverted (d = 7.3 nm) (II, III, and IV) QW structures as a function of the gate voltage measured for B = 0 T at T = 30 mK. The device sizes are (20.0 × 13.3) μm^2 for devices I and II, (1.0 × 1.0) μm^2 for device III, and (1.0 × 0.5) μm^2 for device IV. The inset shows $R_{14,23}(V_g)$ of two samples from the same wafer, having the same device size (III) at 30 mK (green) and 1.8 K (black) on a linear scale.



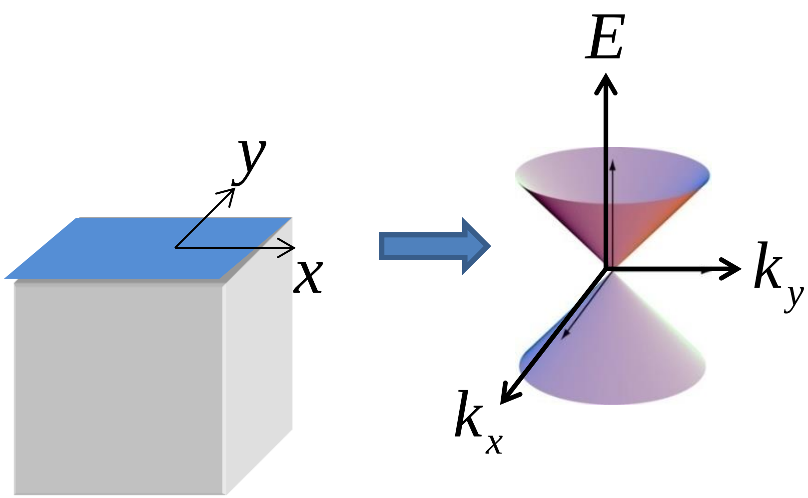
3 dimensional Z_2 Topological insulator

- Band insulator

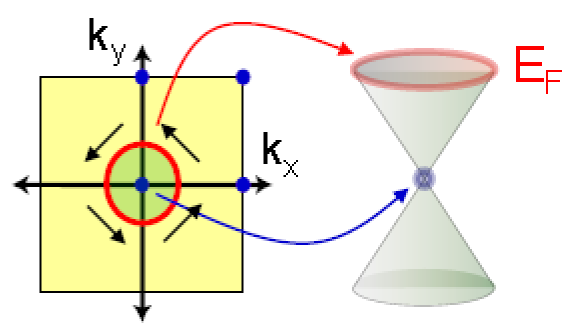


Z_2 topologically nontrivial

- Metallic surface: massless Dirac fermions



an odd number of Dirac cones/surface



Theoretical Predictions made by:
Fu, Kane, & Mele (2007)
Moore & Balents (2007)
Roy (2007)

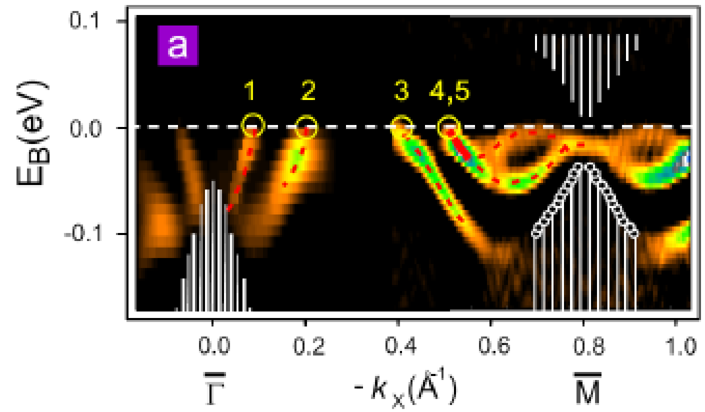
Experimental confirmation

- $Bi_{1-x}Sb_x$ 0.09<x<0.18 theory: Fu & Kane (PRL 2007)

exp: Angle Resolved Photo Emission Spectroscopy

Princeton group (Hsieh et al., Nature 2008)

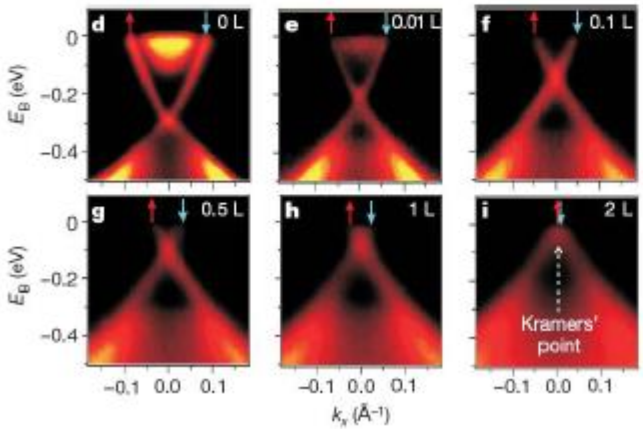
5 surface bands cross Fermi energy

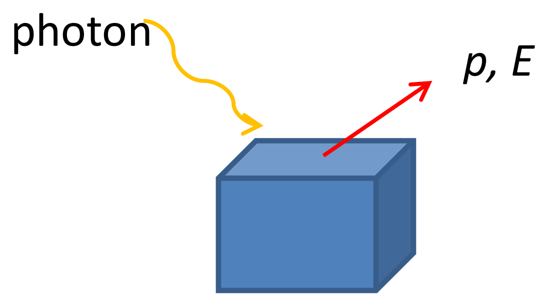


- Bi_2Se_3

ARPES exp.: Xia et al., Nature Phys. 2009

a single Dirac cone





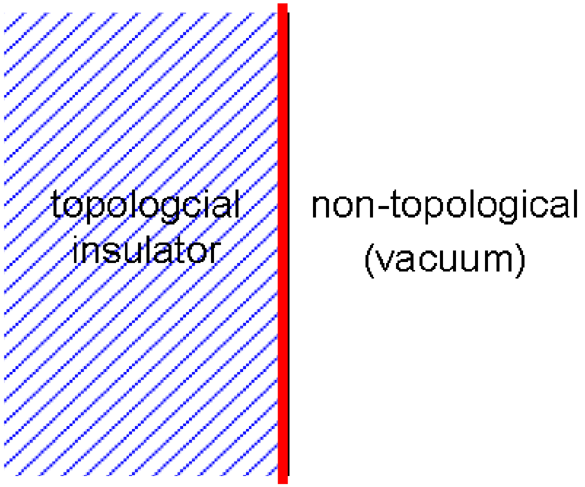
Other topological insulators:

Bi_2Te_3, Bi_2Te_2Se, …

Topological superconductors

- BCS superconductors with a fully gapped Fermi surface
- characterized by a topological number
- gapless excitations at boundaries (Dirac or Majorana)

stable



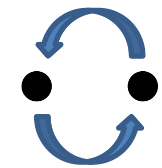
Examples: p+ip superconductor, ^{3}He, …

particle-hole symmetry (BdG Hamiltonian)

2D p+ip superconductor

^{3}He-A thin film, Sr_2RuO_4

- (p_x+ip_y)-wave Cooper pairing



- Hamiltonian

Nambu-spinor $\begin{pmatrix} c_{\vec{p}} \\ c^\dagger_{-\vec{p}} \end{pmatrix}$ (spinless fermions)

$$H_{\vec{p}} = \begin{pmatrix} \frac{p^2}{2m} - \mu & \frac{\Delta}{p_F}\left(p_x + ip_y\right) \\ \frac{\Delta}{p_F}\left(p_x - ip_y\right) & \mu - \frac{p^2}{2m} \end{pmatrix} = \vec{d}\left(\vec{p}\right)\cdot\vec{\sigma}$$

$$\hat{d} = \vec{d}\big/\left|\vec{d}\right|$$

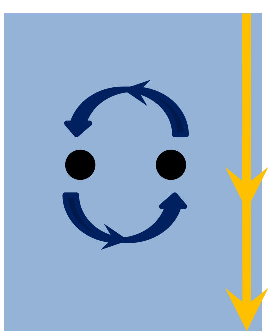
$$S^2$$

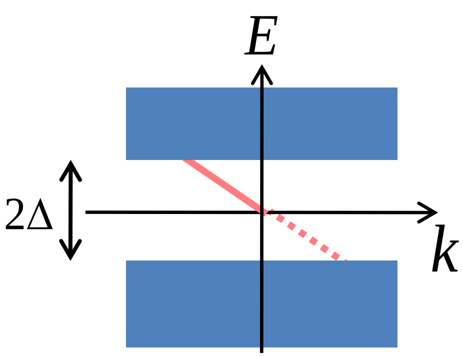
$$\left(p_x, p_y\right) \mapsto S^2$$

wrapping # = 1

- Majorana edge state

p_x+ip_y



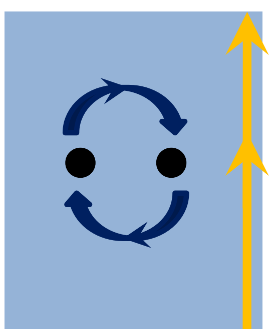


$$\gamma_k = \gamma^\dagger_{-k}$$

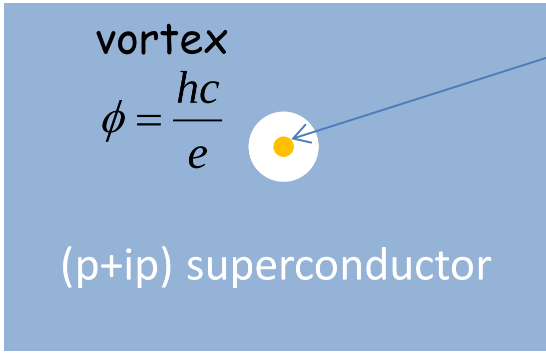
$$\psi\left(x\right) = \int_{k>0} \left(e^{ikx}\gamma_k + e^{-ikx}\gamma^\dagger_k\right) dk$$

$$= \psi^\dagger\left(x\right)$$

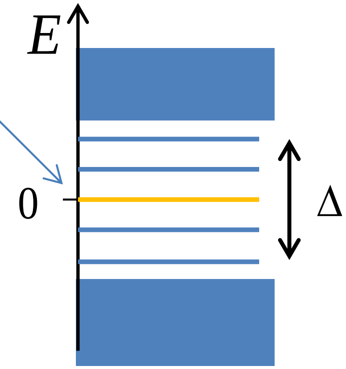
p_x-ip_y



Majorana zeromode in a quantum vortex



Zero-energy Majorana bound state



energy spectrum near a vortex

zero mode $\varepsilon_0 = 0$ $\gamma_0 = \gamma_0^+$

Majorana fermion

If there are 2N vortices, then the ground-state degeneracy = 2^N.

1D p-wave superconductor (Kitaev 2000)

P-wave SC

Majorana fermion

Q: How many classes of topological insulators/superconductors exist in nature?

Topological insulators/superconductors should be stable against arbitrary perturbations (deformation of Hamiltonian) that respect symmetry constraints.

classification based on generic symmetries:

time reversal

charge conjugation (particle hole) SC

random matrix theory

A: There are 5 classes of TPIs or TPSCs in each spatial dimension.

$3Z$ & $2Z_2$

Table of topological insulators/superconductors for d=1,2,3

	10 Symmetry Classes	TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	Z	--
	AI (orthogonal)	+1	0	0	--	--	--
	AII (symplectic)	-1	0	0	--	Z_2	Z_2
Chiral	AIII (chiral unitary)	0	0	1	Z	--	Z
	BDI (chiral orthogonal)	+1	+1	1	Z	--	--
	CII (chiral symplectic)	-1	-1	1	Z	--	Z_2
BdG	D (p-wave SC)	0	+1	0	Z_2	Z	--
	C (d-wave SC)	0	-1	0	--	Z	--
	DIII (p-wave TRS SC)	-1	+1	1	Z_2	Z_2	Z
	CI (d-wave TRS SC)	+1	-1	1	--	--	Z

Altland & Zirnbauer, PRB (1997)

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

Table of topological insulators/superconductors for d=1,2,3

	10 Symmetry Classes	TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	Z (IQHE)	--
	AI (orthogonal)	+1	0	0	--	--	--
	AII (symplectic)	−1	0	0	--	Z_2 (QSHE)	Z_2 (Z_2TPI)
Chiral	AIII (chiral unitary)	0	0	1	Z	--	Z
	BDI (chiral orthogonal)	+1	+1	1	Z (polyacetylene (SSH))	--	--
	CII (chiral symplectic)	−1	−1	1	Z	--	Z_2
BdG	Majorana → D (p-wave SC)	0	+1	0	Z_2 (p SC)	Z (p+ip SC)	--
	C (d-wave SC)	0	−1	0	--	Z (d+id SC)	--
	Majorana → DIII (p-wave TRS SC)	−1	+1	1	Z_2	Z_2 ((p+ip)x(p-ip) SC)	Z (^{3}He-B)
	CI (d-wave TRS SC)	+1	−1	1	--	--	Z

Altland & Zirnbauer, PRB (1997)

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

Periodic table of topological insulators/superconductors

	d												
Cartan	0	1	2	3	4	5	6	7	8	9	10	11	...
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

period $d = 2$

period $d = 8$

A. Kitaev, AIP Conf. Proc. 1134, 22 (2009); arXiv:0901.2686 K-theory, Bott periodicity

Ryu, Schnyder, AF, Ludwig, NJP 12, 065010 (2010) massive Dirac Hamiltonian

M. Stone, C.-K. Chiu, A. Roy, J. Phys. A 44, 045001 (2011) representation of Clifford algebras

Table of topological insulators/superconductors for d=1,2,3

	10 Symmetry Classes	TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	Z	--
	AI (orthogonal)	+1	0	0	--	--	--
	AII (symplectic)	-1	0	0	--	Z_2	Z_2
Chiral	AIII (chiral unitary)	0	0	1	Z	--	Z
	BDI (chiral orthogonal)	+1	+1	1	Z	--	--
	CII (chiral symplectic)	-1	-1	1	Z	--	Z_2
BdG	D (p-wave SC)	0	+1	0	Z_2	Z	--
	C (d-wave SC)	0	-1	0	--	Z	--
	DIII (p-wave TRS SC)	-1	+1	1	Z_2	Z_2	Z
	CI (d-wave TRS SC)	+1	-1	1	--	--	Z

Altland & Zirnbauer, PRB (1997)

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

Time-reversal operator

$$H = \sum_{i,j} c_i^{\dagger} H_{ij} c_j$$

Spin 0 case $T = \mathrm{K}$ $T:\ H_{ij} \rightarrow T H_{ij} T^{-1} = H_{ij}^{*}$

Complex conjugation $T^2 = 1$ integer Spin

Spin ½ case $T = i\sigma_y \mathrm{K}$ $T:\ H_{ij} \rightarrow T H_{ij} T^{-1} = \sigma_y H_{ij}^{*} \sigma_y$

$T^2 = -1$

Classification of Hamiltonian in terms of time-reversal symmetry

$$\text{TRS} = \begin{cases} +1 & \text{if } THT^{-1} = H \text{ and } T^2 = +1 \\ -1 & \text{if } THT^{-1} = H \text{ and } T^2 = -1 \\ 0 & \text{if } THT^{-1} \neq H \end{cases}$$

Table of topological insulators/superconductors

		TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	Z	--
	AI (orthogonal)	+1	0	0	--	--	--
	AII (symplectic)	-1	0	0	--	Z_2	Z_2
Chiral	AIII (chiral unitary)	0	0	1	Z	--	Z
	BDI (chiral orthogonal)	+1	+1	1	Z	--	--
	CII (chiral symplectic)	-1	-1	1	Z	--	Z_2
BdG	D (p-wave SC)	0	+1	0	Z_2	Z	--
	C (d-wave SC)	0	-1	0	--	Z	--
	DIII (p-wave TRS SC)	-1	+1	1	Z_2	Z_2	Z
	CI (d-wave TRS SC)	+1	-1	1	--	--	Z

Particle-hole transformation for Bogoliubov-de Gennes Hamiltonian

Examples:

(1) spinless p_x+ip_y

$$H=\frac{1}{2}\sum_{\vec{k}}\begin{pmatrix}c_{\vec{k}}^{\dagger} & c_{-\vec{k}}\end{pmatrix} H_{\vec{k}}\begin{pmatrix}c_{\vec{k}}\\ c_{-\vec{k}}^{\dagger}\end{pmatrix}$$

$$H_{\vec{k}}=\begin{pmatrix}\varepsilon_{\vec{k}} & \Delta\left(k_x-ik_y\right)\\ \Delta\left(k_x+ik_y\right) & -\varepsilon_{-\vec{k}}\end{pmatrix}=\Delta\left(k_x\tau_x+k_y\tau_y\right)+\varepsilon_k\tau_z$$

Particle-hole symmetry $\tau_x H_{-\vec{k}}^{*}\tau_x=-H_{\vec{k}}$ $C=\tau_x \mathrm{K}$ $C^2=1$

$$E_n\rightarrow -E_n$$

$$\begin{pmatrix}u_n\\ v_n\end{pmatrix}\rightarrow\begin{pmatrix}v_n^{*}\\ u_n^{*}\end{pmatrix}$$

$$\begin{pmatrix}\psi\\ \psi^{\dagger}\end{pmatrix}=\sum_{E_n>0}\left[\begin{pmatrix}u_n\\ v_n\end{pmatrix}a_n+\begin{pmatrix}v_n^{*}\\ u_n^{*}\end{pmatrix}a_n^{\dagger}\right]+\begin{pmatrix}u_0\\ v_0\end{pmatrix}\gamma_0$$

$$\gamma_0=\gamma_0^{\dagger}$$

Majorana fermion

$$u_0=v_0^{*}$$

Particle-hole transformation for Bogoliubov-de Gennes Hamiltonian

(2) $d_{x2-y2}+id_{xy}$ (spin singlet pairing)

$$H=\sum_{\vec{k}}\begin{pmatrix} c_{k\uparrow}^{\dagger} & c_{-k\downarrow} \end{pmatrix} H_k \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^{\dagger} \end{pmatrix}$$

$$H_{\vec{k}}=\begin{pmatrix} \varepsilon_{\vec{k}} & \Delta\left(k_x^2-k_y^2-ik_xk_y\right) \\ \Delta\left(k_x^2-k_y^2+ik_xk_y\right) & -\varepsilon_{-\vec{k}} \end{pmatrix}$$

$$=\Delta\left[\left(k_x^2-k_y^2\right)\tau_x+k_xk_y\tau_y\right]+\varepsilon_k\tau_z$$

Particle-hole symmetry $\tau_y H_{-\vec{k}}^{*}\tau_y=-H_{\vec{k}}$ $C=i\tau_y\mathrm{K}$ $C^2=-1$

$$E_n\to -E_n$$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}\to\begin{pmatrix} v_n^{*} \\ -u_n^{*} \end{pmatrix}$$

$$\begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^{\dagger} \end{pmatrix}=\sum_{E_n>0}\left[\begin{pmatrix} u_n \\ v_n \end{pmatrix}a_{n\uparrow}+\begin{pmatrix} v_n^{*} \\ -u_n^{*} \end{pmatrix}a_{n\downarrow}^{\dagger}\right]$$

No Majorana

Classification of Hamiltonian in terms of particle-hole symmetry

$$\text{PHS} = \begin{cases} +1 & \text{if } C^{-1}HC = -H \text{ and } C^2 = +1 \\ -1 & \text{if } C^{-1}HC = -H \text{ and } C^2 = -1 \\ 0 & \text{if } C^{-1}HC \neq H \end{cases}$$

Table of topological insulators/superconductors

		TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	Z	--
	AI (orthogonal)	+1	0	0	--	--	--
	AII (symplectic)	-1	0	0	--	Z_2	Z_2
Chiral	AIII (chiral unitary)	0	0	1	Z	--	Z
	BDI (chiral orthogonal)	+1	+1	1	Z	--	--
	CII (chiral symplectic)	-1	-1	1	Z	--	Z_2
BdG	D (p-wave SC)	0	+1	0	Z_2	Z	--
	C (d-wave SC)	0	-1	0	--	Z	--
	DIII (p-wave TRS SC)	-1	+1	1	Z_2	Z_2	Z
	CI (d-wave TRS SC)	+1	-1	1	--	--	Z

"Chiral symmetry" (CS)

There is a unitary operator which anticommutes with Hamiltonian.

$$H\Gamma + \Gamma H = 0$$

$$H = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 1: lattice model with hopping between AB sublattices only

$$H = \sum_{\substack{a\in A \\ b\in B}} \left(t_{ab} c_a^\dagger c_b + t_{ab}^* c_b^\dagger c_a \right)$$

Example 2: time-reversal × particle-hole (T and C are antiunitary)

$$THT^{-1} = H$$

$$CHC^{-1} = -H$$

$$\Longrightarrow \quad TCHC^{-1}T^{-1} = -H \qquad TCH = -HTC$$

Classification of free-fermion Hamiltonian in terms of generic discrete symmetries

- Time-reversal symmetry (TRS)

$$THT^{-1} = H$$

$$\mathrm{TRS} = \begin{cases} 0 & \text{no TR invariance} \\ +1 & T^2 = +1 \\ -1 & T^2 = -1 \end{cases}$$

spin 0

spin 1/2

- Particle-hole symmetry (PHS)

BdG Hamiltonian

$$CHC^{-1} = -H$$

$$\mathrm{PHS} = \begin{cases} 0 & \text{no PH invariance} \\ +1 & C^2 = +1 \\ -1 & C^2 = -1 \end{cases}$$

Singlet SC

$$\mathrm{TRS} = \mathrm{PHS} = 0,\ \mathrm{CS} = 1$$

$$3 \times 3 + 1 = 10$$

トポロジカル絶縁体・超伝導体の分類表の導出

- 表面状態のAnderson非局在
  - Nonlinear sigma model with a topological term
- Dirac Hamiltonian
  - Clifford代数の表現論（K理論）
  - Dimensional reduction

Anderson delocalization of boundary states

- Gapless boundary modes are topologically protected.
- They are stable against any local perturbation.

  (respecting discrete symmetries)
- They should **never** be Anderson localized by **disorder**.

Nonlinear sigma models for Anderson localization

of gapless boundary modes

$$S = \int d^{d-1}r \, \mathrm{tr}\left(\partial Q\right)^2 + \text{topological term}$$ (with no adjustable parameter)

$Q \in M$

Z_2 top. term $\quad \pi_{d-1}(M) = Z_2$

WZW term $\quad \pi_d(M) = Z$

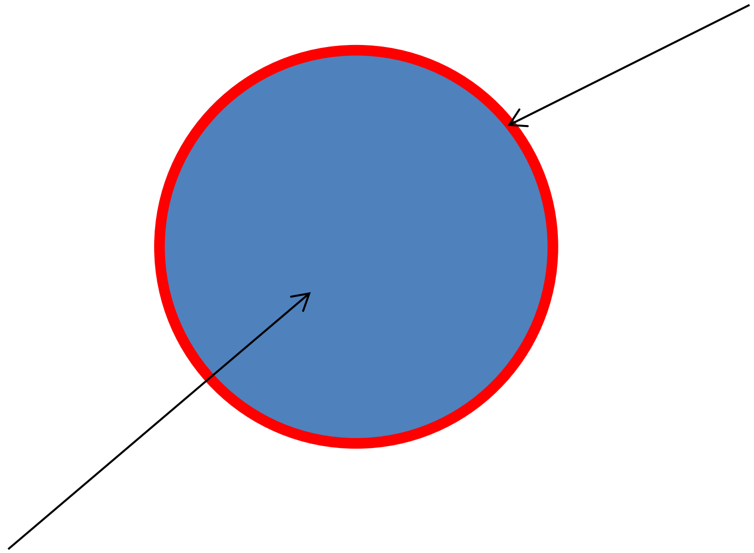
~~θ-term~~

bulk: d dimensions

boundary: d -1 dimensions

bulk-boundary correspondence

Anderson delocalization
topologically stable, gapless excitations



Topological insulator/superconductor

fully gapped (no excitations)

Nonlinear sigma model: (Wegner, Efetov, Altshuler-Kravtsov-Lerner, …)

low-energy effective field theory for Nambu-Goldstone bosons

	$E = \int (\partial \vec{n})^2 dx d\tau + i2\pi SN$	$E = \int \mathrm{tr}(\partial Q)^2 d^2 r + i\theta N$
	Antiferromagnets	Integer Quantum Hall effect
N-G bosons	magnons	Diffuson
Ordered phase	antiferromagnetic	metallic
Disordered phase	paramagnetic	insulating
Order parameter	$\vec{n} \in R^3 \quad \vec{n} \cdot \vec{n} = 1$	$Q \in \mathrm{U}(2N) \quad Q \approx \mathrm{diag}(1_N, -1_N)$
Target space	$G/H = \mathrm{O}(3)/\mathrm{O}(2)$	$G/H = U(2N)/U(N) \times U(N)$
	$\pi_2(G/H) = Z$	$\pi_2(G/H) = Z$
	Haldane	Pruisken

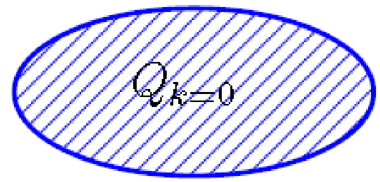
Topological terms lead to nonperturbative effects.

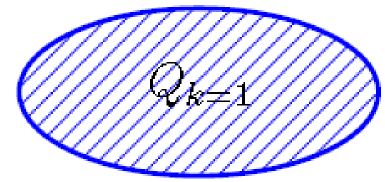
S = 1/2 ($\theta = \pi$) のとき、massless (critical)

Nonlinear sigma model

Symplectic class (AII)

N-G bosons	Diffuson & Cooperon
Ordered phase	metallic
Disordered phase	insulating





Matrix fields $\quad Q^2 = 1_{4N}, \quad Q^T = Q, \quad \mathrm{Tr}\, Q = 0$

Target space $\quad G/H = \mathrm{O(4N)/O(2N)} \times \mathrm{O(2N)}$

$$\pi_2 (G/H) = Z_2$$

Fendley, PRB (2001)

2 distinct sectors in the space of field configurations

$$e^{-S_1} + e^{-S_2} \quad \text{or} \quad e^{-S_1} - e^{-S_2}$$

(no top. term) $\qquad$ (with Z_2 top. term)

常に metallic

NLSM topological terms $\pi_d(G/H)$

complex case:

	$G/H \setminus d$	$d=0$	$d=1$	$d=2$	$d=3$
A	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	$\mathrm{U}(N)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

real case:

	$G/H \setminus d$	$d=0$	$d=1$	$d=2$	$d=3$
AI	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\mathbb{Z}$	0	0	0
BDI	$\mathrm{U}(2N)/\mathrm{Sp}(N)$	0	$\mathbb{Z}$	0	0
D	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
DIII	$\mathrm{O}(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
AII	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
CII	$\mathrm{U}(N)/\mathrm{O}(N)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	$\mathrm{Sp}(N)/\mathrm{U}(N)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
CI	$\mathrm{Sp}(N)$	0	0	0	$\mathbb{Z}$

Z_2: a Z_2 topological term can exist in d dimensions

Z: a WZW term can exist in d-1 dimensoins

Periodic table of topological insulators/superconductors

	d												
Cartan	0	1	2	3	4	5	6	7	8	9	10	11	…
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

period d = 2

period d = 8

A. Kitaev, AIP Conf. Proc. 1134, 22 (2009); arXiv:0901.2686 K-theory, Bott periodicity

Ryu, Schnyder, AF, Ludwig, NJP 12, 065010 (2010) massive Dirac Hamiltonian

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Classification of Dirac mass

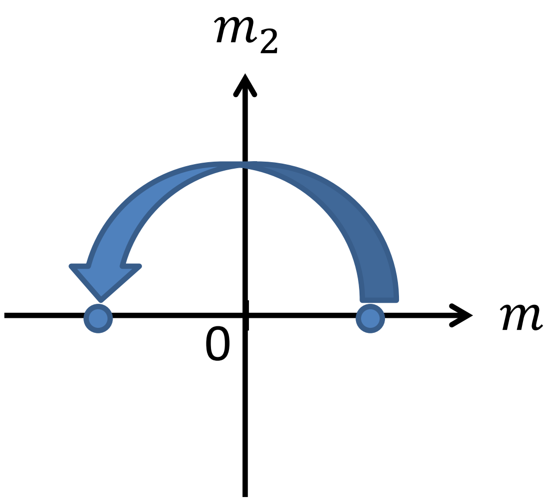
$$H = \sum_{\mu=1}^{d} k_\mu \gamma_\mu + m\gamma_0 \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$$

If $m\gamma_0$ is a unique Dirac mass, then gapped phases with opposite sign of m are topologically distinct phases.

Examples

(1) $d = 2$ class A (IQHE)

$$H = k_x\sigma_x + k_y\sigma_y + m\sigma_z$$



(2) $d = 1$ class A

$$H = k_x\sigma_x + m_1\sigma_y + m_2\sigma_z$$

Two gapped states ($m_1 > 0$ and $m_1 < 0$) are connected without closing a gap.

(3) $d = 1$ class AIII $\quad \{H, \sigma_z\} = 0$

$$H = k_x\sigma_x + m\sigma_y$$

$m\sigma_y$ is a unique mass term.

Set of possible mass terms: classifying space

Example: $d = 2$ class A (IQHE)

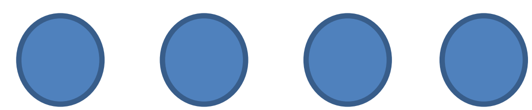
$$H = k_x \underbrace{\sigma_x \otimes 1_N}_{\gamma_1} + k_y \underbrace{\sigma_y \otimes 1_N}_{\gamma_2} + \gamma_0 \qquad \{\gamma_a, \gamma_b\} = 2\delta_{a,b}$$

$$\gamma_0 = \sigma_z \otimes A \qquad A = U \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix} U^{\dagger} \qquad (N = n + m)$$

$$\gamma_0 \Longleftrightarrow U \in \frac{U(n+m)}{U(n) \times U(m)}$$

Classifying space C_0
= Complex Grassmanian

$$\pi_0 \left[\oplus_{m,n} U(m+n)/U(m) \times U(n) \right] = \mathbb{Z}$$

···  ···

There are topologically distinct gapped phases labelled by an integer index.

The parameter n corresponds to Chern number.

$$H = k_x \sigma_x + k_y \sigma_y + \left(\varepsilon - k^2\right) \sigma_z$$

Chern # = $\begin{cases} 1 & (\varepsilon > 0) \\ 0 & (\varepsilon < 0) \end{cases}$

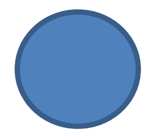
Example: $d = 1$ class A (no symmetry constraint)

$$H = k_x \sigma_z \otimes 1_N + \gamma_0$$

$$\gamma_0 = \begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix} \qquad U \in U(N) \qquad \text{Classifying space } C_1$$

$$\pi_0\left(U(N)\right) = 0$$

There is only a single gapped phase.



まとめ

- アンダーソン局在
  - スケーリング理論、空間次元、対称性、トポロジー
  - ユニバーサリティー・クラス（10種類のランダム行列集団）
  - 臨界現象の理論は未発達（$d \geq 2$）
- トポロジカル絶縁体・超伝導体
  - Z あるいは Z_2 のトポロジカル数で分類されるバンド絶縁体・BCS超伝導体
  - 周期表: 各空間次元で10種類の対称類のうち、3種類がZ、2種類がZ_2

- 話せなかった最近の話題
  - 結晶対称性（鏡映、C_3、etc）と時間反転対称性のもとでトポロジカルに安定な絶縁体 topological crystalline insulators: SnTe
  - 相互作用する系（fermions, bosons (spins)）のトポロジカル相
    - エンタングルメントは短距離 : symmetry protected topological (SPT) phase
      X.-G. Wenら、Fidkowski-Kitaev、Pollmann-Berg-Turner、Lu-Vishwanath、,,,
      １次元 : 行列積状態 (MPS) AKLT state group cohomology
    - エンタングルメントが長距離 : トポロジカル秩序 X.-G. Wen
      基底状態の縮退度が系のトポロジーに依存 分数量子系とその一般化