KPZ 普遍性の新たな展開

笹本智弘（東工大）

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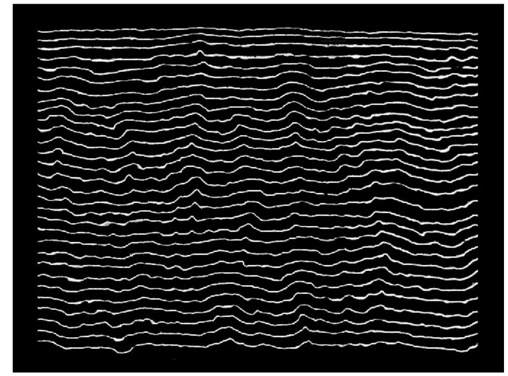
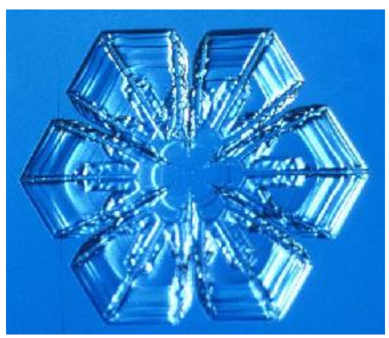
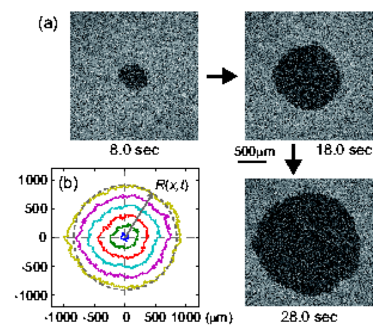
1. KPZ for surface growth



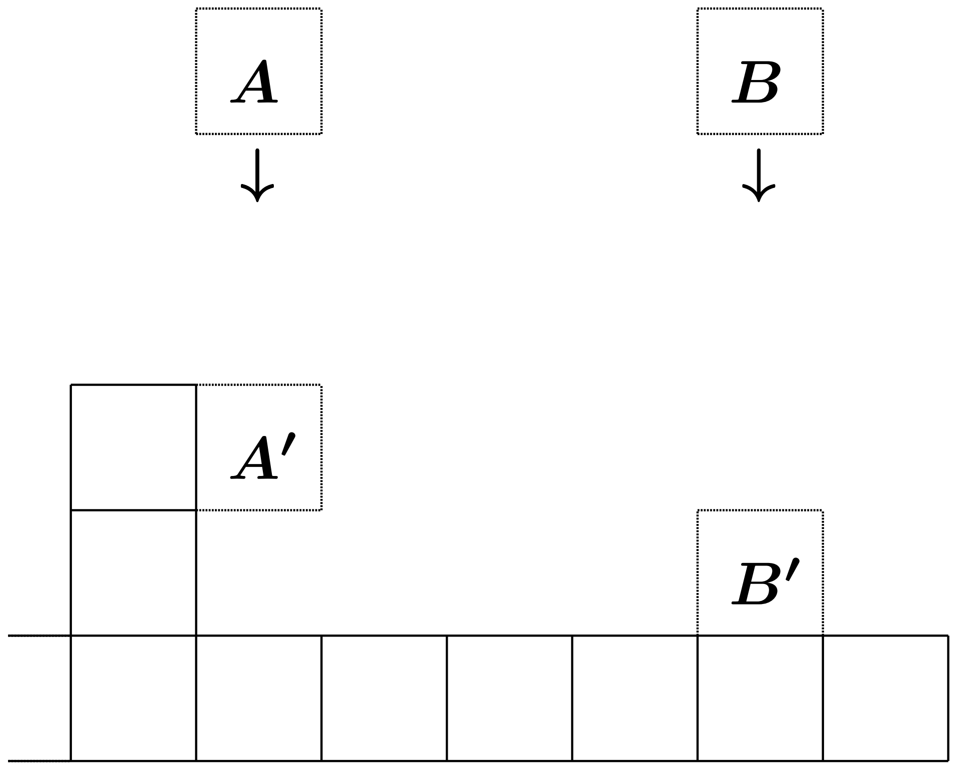
- Paper combustion, bacteria colony, crystal growth, liquid crystal turbulence
- Non-equilibrium statistical mechanics
- Connections to integrable systems

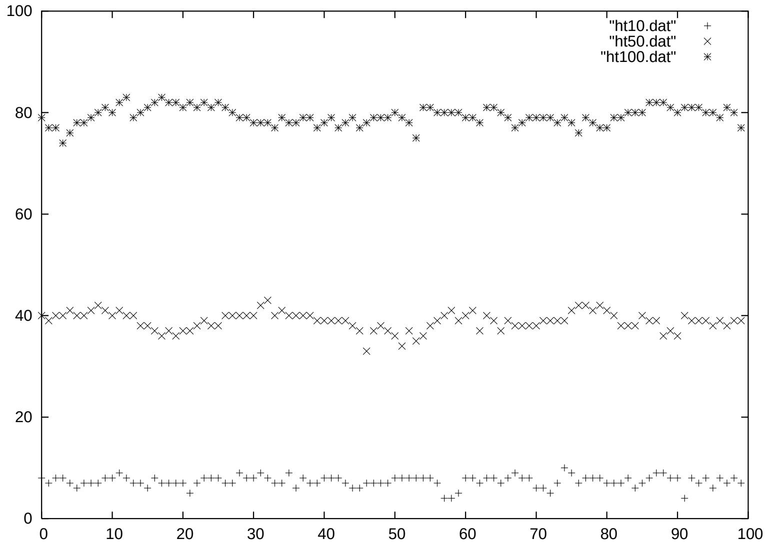




Simulation models

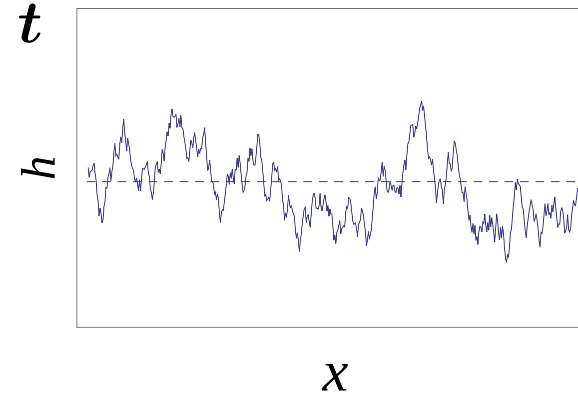
Ex: ballistic deposition





Scaling

$h(x,t)$: surface height at position x and at time t



Scaling (L: system size)

$$W(L,t) = \langle (h(x,t) - \langle h(x,t) \rangle)^2 \rangle^{1/2}$$
$$= L^{\alpha} \Psi(t/L^{z})$$

For $t \to \infty$ $\quad W(L,t) \sim L^{\alpha}$

For $t \sim 0$ $\quad W(L,t) \sim t^{\beta}$ where $\alpha = \beta z$

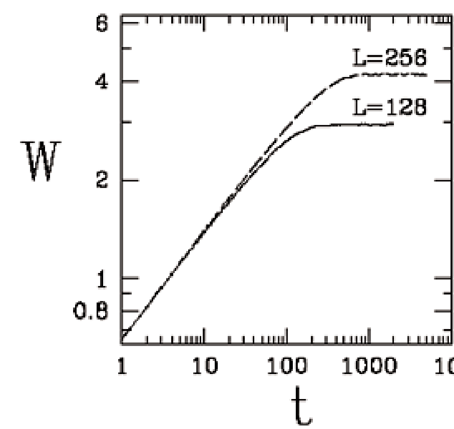


Figure 1. Interface width W versus time t for the RS
(Ref. [11]) in $1+1$ dimensions, in two different lattice

In many models, $\alpha = 1/2, \beta = 1/3$

Dynamical exponent $z = 3/2$: Anisotropic scaling

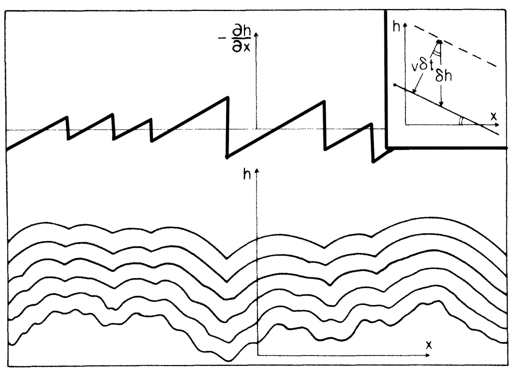
KPZ equation

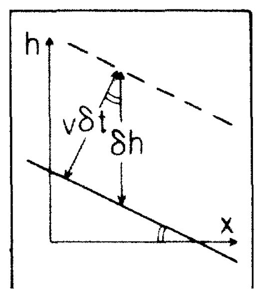
1986 Kardar Parisi Zhang

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where η is the Gaussian noise with covariance

$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$



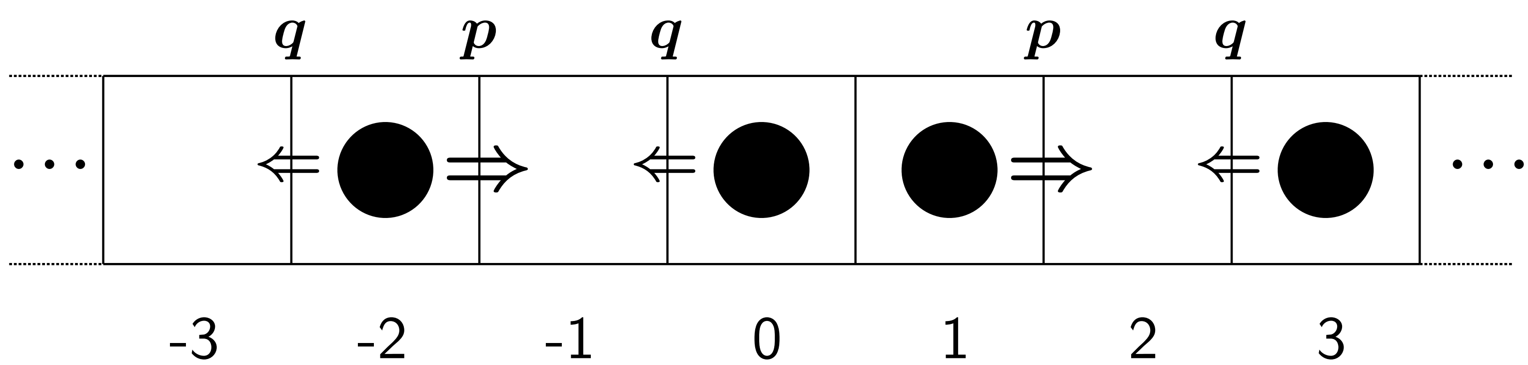


$$\begin{aligned}\partial_t h &= v\sqrt{1+(\partial_x h)^2} \\ &\simeq v + (v/2)(\partial_x h)^2 + \ldots\end{aligned}$$

- Dynamical RG analysis: $\rightarrow \alpha = 1/2, \beta = 1/3$(KPZ class)
- New analytic and experimental developments

2: Limiting height distribution

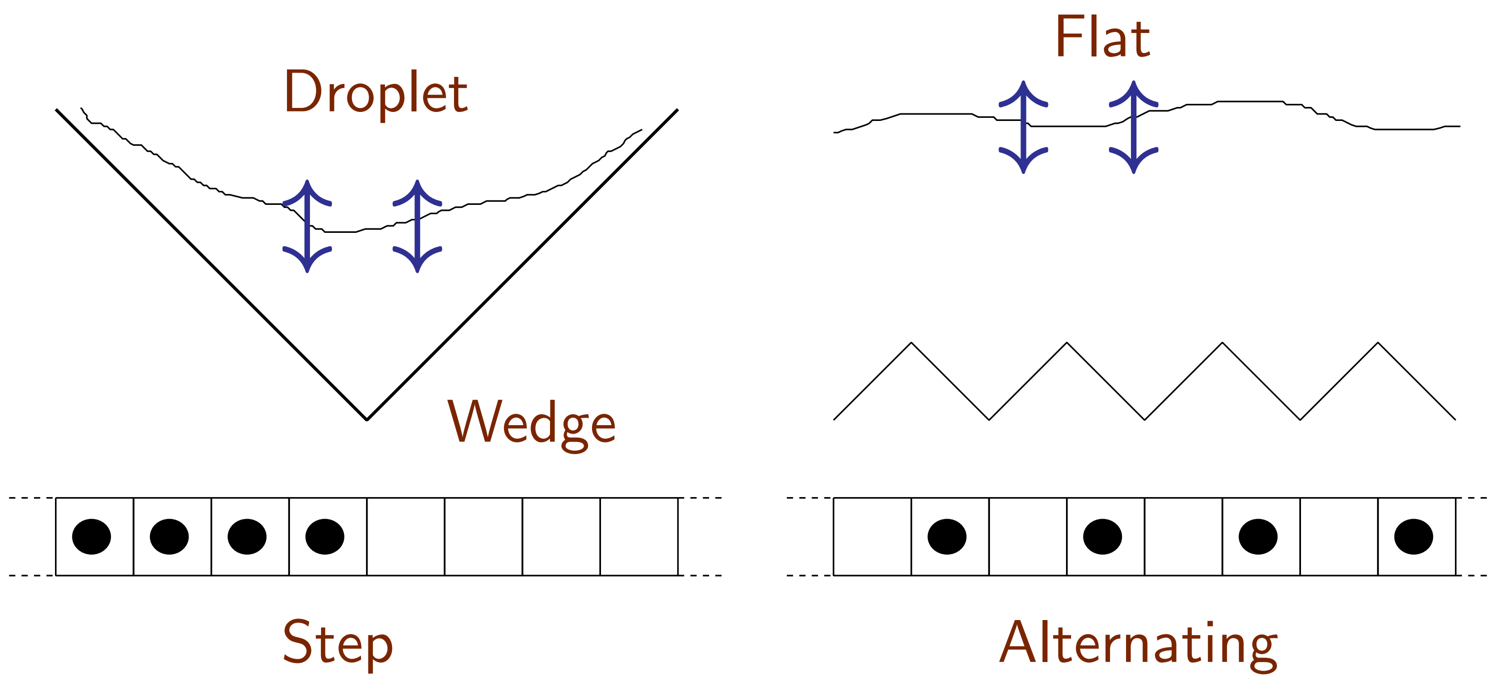
ASEP = asymmetric simple exclusion process



- TASEP(Totally ASEP, $p = 0$ or $q = 0$)
- $N(x, t)$: Integrated current at $(x, x + 1)$ upto time t
- Bernoulli (each site is independently occupied with probability ρ) is stationary

Mapping to surface growth

2 initial conditions besides stationary



Integrated current $N(x,t)$ in ASEP

$\Leftrightarrow$ Height $h(x,t)$ in surface growth

TASEP with step i.c.

2000 Johansson

As $t \to \infty$

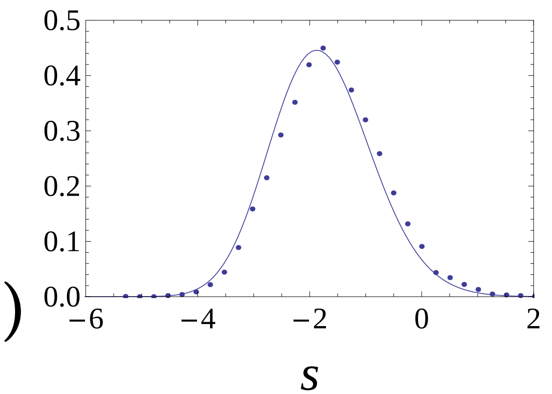
$$N(0,t) \simeq \frac{1}{4}t - 2^{-4/3}t^{1/3}\xi_2$$

Here $N(x=0,t)$ is the integrated current of TASEP at the origin and ξ_2 obeys the GUE Tracy-Widom distribution;

$$F_2(s) = \mathbb{P}[\xi_2 \le s] = \det(1 - P_s K_{\mathrm{Ai}} P_s)$$

where P_s: projection onto the interval $[s,\infty)$ and K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$



Random universality in KPZ universality

Tracy-Widom distributions

Random matrix theory, Gaussian ensembles

H: $N \times N$ matrix

$$P(H)dH = \frac{1}{Z_{N\beta}} e^{-\frac{\beta}{2}\mathrm{Tr}H^2}$$

GOE(real symmetric, $\beta = 1$), **GUE**(hermitian, $\beta = 2$).

Joint eigenvalue distribution

$$P_{N\beta}(x_1, x_2, \ldots, x_N) = \frac{1}{Z_{N\beta}} \prod_{1 \leq i < j \leq N} (x_i - x_j)^\beta \prod_{i=1}^{N} e^{-\frac{\beta}{2}x_i^2}$$

- Average density $\cdots$ Wigner semi-circle

Largest eigenvalue distribution

Largest eigenvalue distribution of Gaussian ensembles

$$\mathbb{P}_{N\beta}[x_{\max} \leq s] = \frac{1}{Z_{N\beta}} \int_{(-\infty,s]^N} \prod_{i<j} (x_i - x_j)^{\beta} \prod_i e^{-\frac{\beta}{2}x_i^2} dx_1 \cdots dx$$

Scaling limit (expected to be universal)

$$\lim_{N\to\infty} \mathbb{P}_{N\beta}\left[(x_{\max} - \sqrt{2N})\sqrt{2}N^{1/6} < s\right] = F_{\beta}(s)$$

GUE (GOE) Tracy-Widom distribution

Tracy-Widom distributions

GUE Tracy-Widom distribution

$$F_2(s) = \det(1 - P_s K_2 P_s)$$

where P_s: projection onto $[s, \infty)$ and K_2 is the Airy kernel

$$K_2(x, y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x + \lambda)\mathrm{Ai}(y + \lambda)$$

Painlevé II representation

$$F_2(s) = \exp\left[-\int_s^\infty (x - s)u(x)^2 dx\right]$$

where $u(x)$ is the solution of the Painlevé II equation

$$\frac{\partial^2}{\partial x^2} u = 2u^3 + xu, \quad u(x) \sim \mathrm{Ai}(x) \quad x \to \infty$$

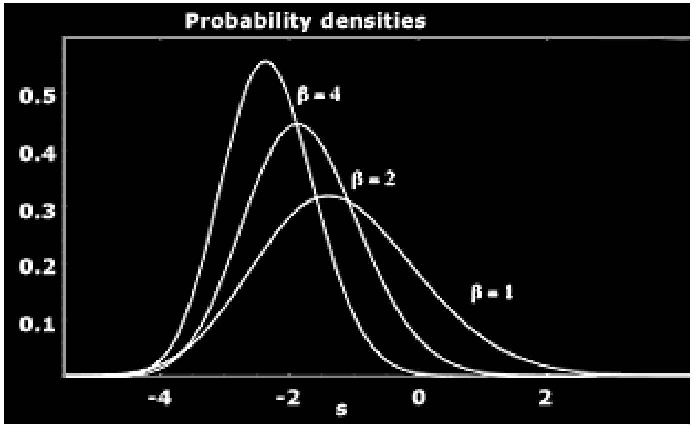
GOE Tracy-Widom distribution

$$\boldsymbol{F_1(s) = \exp\left[-\frac{1}{2}\int_s^\infty u(x)dx\right](F_2(s))^{1/2}}$$

GSE Tracy-Widom distribution

$$\boldsymbol{F_4(s) = \cosh\left[-\frac{1}{2}\int_s^\infty u(x)dx\right](F_2(s))^{1/2}}$$

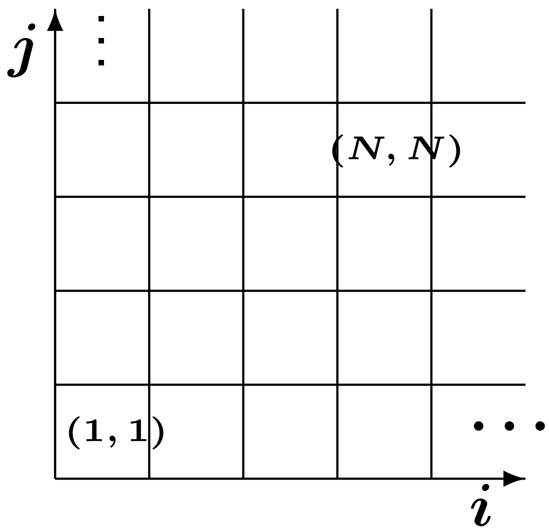
Figures for Tracy-Widom distributions



Step TASEP and random matrix

- Generalize to discrete TASEP with parallel update.

  A waiting time is geometrically distributed.



w_{ij} on (i,j): geometrically distributed waiting time of ith hop of jth particle

- Time at which Nth particle arrives at the origin

$$= \max_{\substack{\text{up-right paths from} \\ (1,1)\text{to}(N,N)}} \left\{ \sum_{(i,j)\text{ on a path}} w_{i,j} \right\} \quad (= G(N,N))$$

Zero temperature directed polymer

LUE formula for TASEP

- By RSK algorithm a matrix of size $N \times N$ with non-negative integer entries is mapped to a pair of semi-standard Young tableau with the same shape λ with entries from $\{1, 2, \ldots, N\}$, with $G(N, N) = \lambda_1$.
- When the jth particle does ith hop with parameter $\sqrt{a_i b_j}$, the measure on λ is given by the Schur measure

$$\frac{1}{Z} s_\lambda(a) s_\lambda(b)$$

- Using a determinant formula of the Schur function and taking the continuous time limit, one gets

$$\mathbb{P}[N(t) \geq N] = \frac{1}{Z_N} \int_{[0,t]^N} \prod_{i<j} (x_i - x_j)^2 \prod_i e^{-x_i} dx_1 \cdots dx_N$$

Generalizations

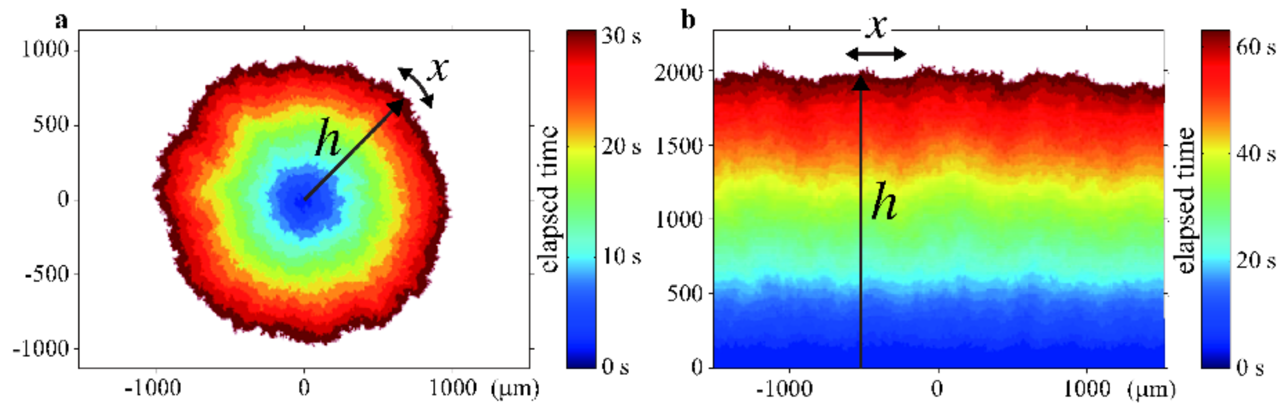
Current Fluctuations of TASEP with flat initial conditions: GOE TW distribution

More generalizations: stationary case: F_0 distribution, multi-point fluctuations: Airy process, etc

Experimental relevance?

What about the KPZ equation itself?

Takeuchi-Sano experiments



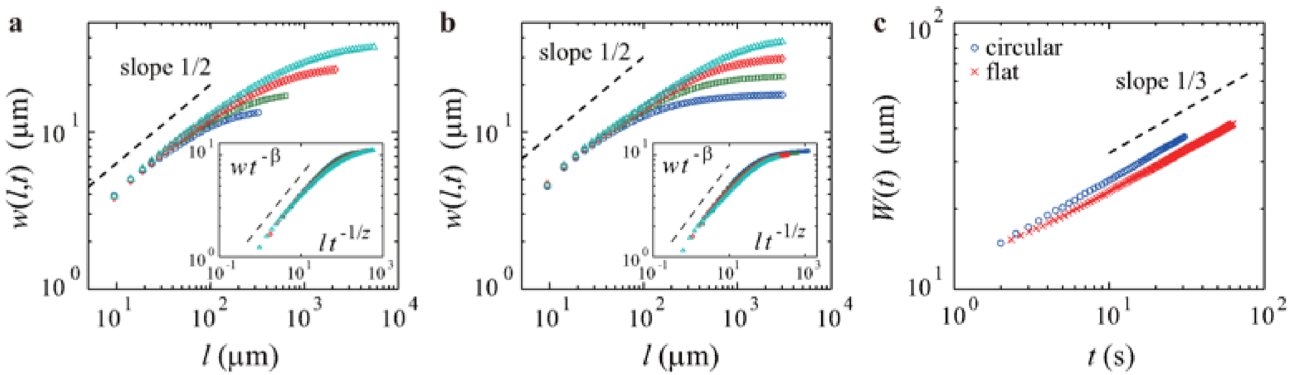


Figure 2 | **Family-Vicsek scaling.** a,b, Interface width $w(l, t)$ against the length scale l at different times t for the circular (a) and flat (b) interfaces. The four data correspond, from bottom to top, to $t = 2.0$ s, 4.0 s, 12.0 s and 30.0 s for the panel a and to $t = 4.0$ s, 10.0 s, 25.0 s and 60.0 s for the panel b. The insets show the same data with the rescaled axes. c, Growth of the overall width $W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$. The dashed lines are guides for the eyes showing the exponent values of the KPZ class.

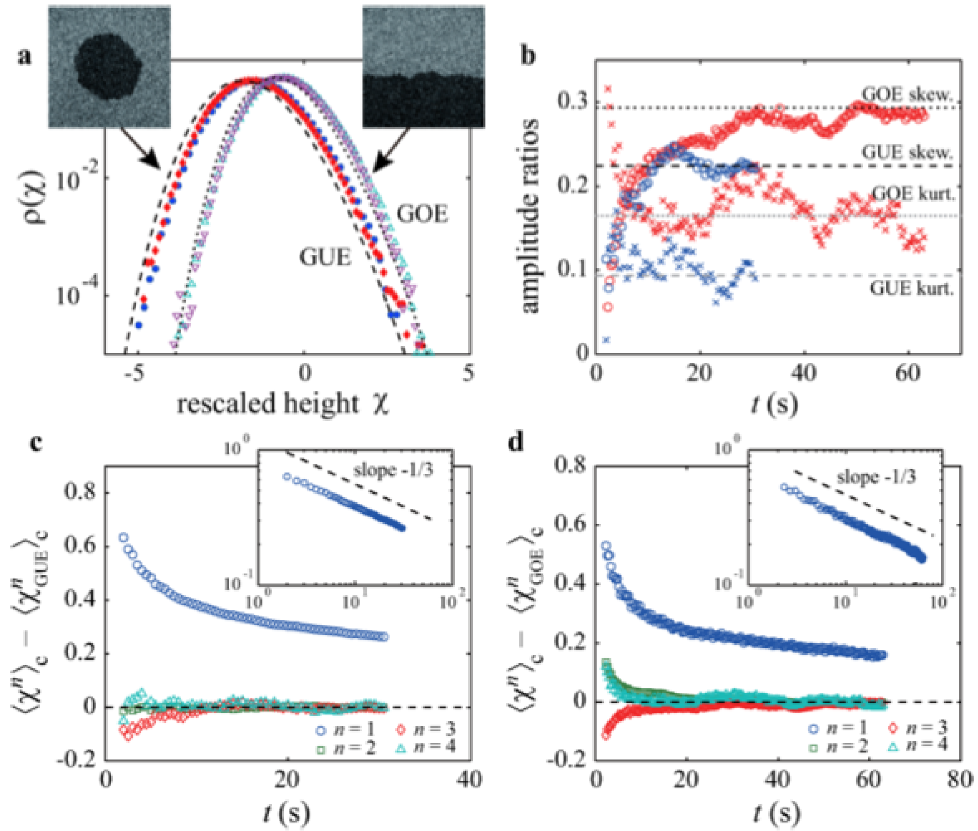


Figure 3 | Universal fluctuations. a, Histogram of the rescaled local height $\chi = (h - v_\infty t)/(\Gamma t)^{1/3}$. The blue and red solid symbols show the histograms for the circular interfaces at $t = 10$ s and 30 s; the light blue and purple open symbols are for the flat interfaces at $t = 20$ s and 60 s, respectively. The dashed and dotted curves show the GUE and GOE TW distributions, respectively. Note that for the GOE TW distribution χ is multiplied by $2^{-2/3}$ in view of the theoretical prediction[31]. b, The skewness (circle) and the kurtosis (cross) of the distribution of the interface fluctuations for the circular (blue) and flat (red) interfaces. The dashed and dotted lines indicate the values of the skewness and the kurtosis of the GUE and GOE TW distributions[31]. c, d, Differences in the cumulants between the experimental data $\langle\chi^n\rangle_c$ and the corresponding TW distributions $\langle\chi^n_{\mathrm{GUE}}\rangle_c$ for the ciruclar interfaces (c) and $\langle\chi^n_{\mathrm{GOE}}\rangle_c$ for the flat interfaces (d). The insets show the same data for $n = 1$ in logarithmic scales. The dashed lines are guides for the eyes with the slope $-1/3$.

See Takeuchi Sano Sasamoto Spohn, Sci. Rep. 1,34(2011)

3. Exact solution for the KPZ equation

Remember the KPZ equation

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

2010 Sasamoto Spohn, Amir Corwin Quastel

- Narrow wedge initial condition
- Based on (i) the fact that the weakly ASEP is KPZ equation (1997 Bertini Giacomin) and (ii) a formula for step ASEP by 2009 Tracy Widom
- The explicit distribution function for finite t

Narrow wedge initial condition

Scalings

$$x \to \alpha^2 x, \quad t \to 2\nu\alpha^4 t, \quad h \to \frac{\lambda}{2\nu} h$$

where $\alpha = (2\nu)^{-3/2}\lambda D^{1/2}$.

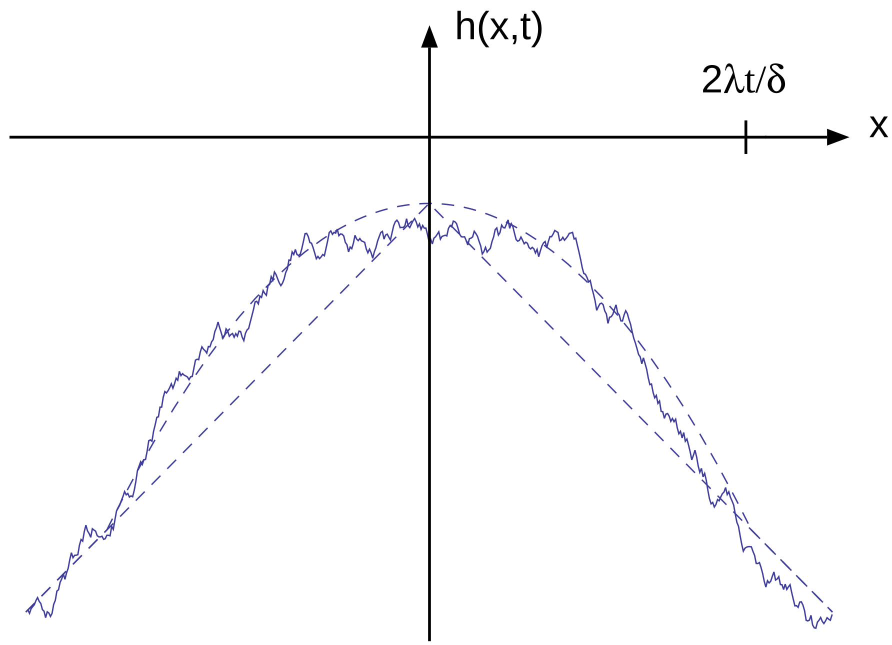
We can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$.

We consider the droplet growth with macroscopic shape

$$h(x,t) = \begin{cases} -x^2/2t & \text{for } |x| \le t/\delta\,, \\ (1/2\delta^2)t - |x|/\delta & \text{for } |x| > t/\delta \end{cases}$$

which corresponds to taking the following narrow wedge initial conditions:

$$h(x,0) = -|x|/\delta\,, \quad \delta \ll 1$$


h(x,t)
2λt/δ
x

Distribution

$$h(x,t) = -x^2/2t - \tfrac{1}{12}\gamma_t^3 + \gamma_t \xi_t$$

where $\gamma_t = (2t)^{1/3}$.

The distribution function of ξ_t

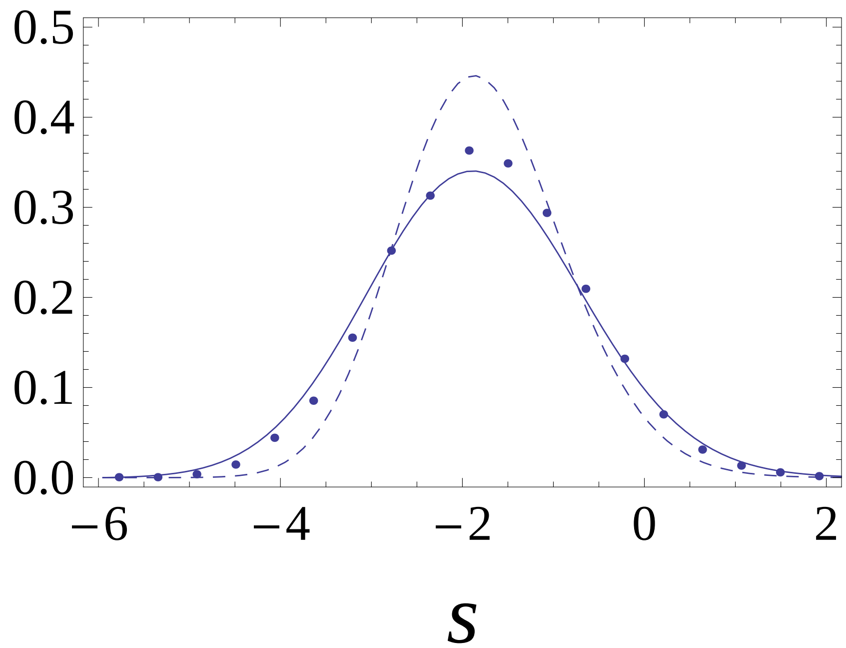
$$F_t(s) = \mathbb{P}[\xi_t \le s] = 1 - \int_{-\infty}^{\infty} \exp\big[- \mathrm{e}^{\gamma_t(s-u)}\big]$$
$$\times \big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\big)\mathrm{d}u$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, P_u is the projection onto $[u,\infty)$ and the kernel B_t is

$$B_t(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t \lambda} - 1}$$

- In the large t limit, F_t tends to F_2.

Finite time KPZ distribution and TW



—: exact KPZ density $F_t'(s)$ at $\gamma_t = 0.94$

– –: Tracy-Widom density ($t \to \infty$ limit)

•: ASEP Monte Carlo at $q = 0.6$, $t = 1024$ MC steps

Cole-Hopf transformation

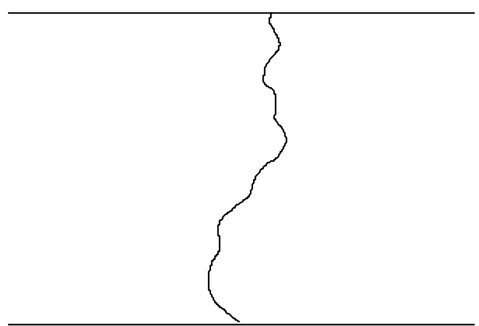
If we set

$$Z(x,t) = \exp\left(h(x,t)\right)$$

this quantity (formally) satisfies

$$\frac{\partial}{\partial t} Z(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t) Z(x,t)$$

This can be interpreted as a (random) partition function for a directed polymer in random environment η.



Replica method

For a system with randomness, e.g. for random Ising model,

$$H = \sum_{\langle ij \rangle} J_{ij} s_i s_j$$

where i is site, $s_i = \pm 1$ is Ising spin, J_{ij} is iid random variable(e.g. Bernoulli), we are often interested in the averaged free energy $\langle \log Z \rangle$, $Z = \sum_{s_i = \pm 1} e^{-H}$.

In some cases, computing $\langle \log Z \rangle$ seems hopeless but the calculation of Nth replica partition function $\langle Z^N \rangle$ is easier.

In replica method, one often resorts to the following identity

$$\langle \log Z \rangle = \lim_{N \to 0} \frac{\langle Z^N \rangle - 1}{N}.$$

For KPZ: Feynmann-Kac and δ Bose gas

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(e^{\int_0^t \eta(b(s),t-s)ds} Z(b(t),0) \right)$$

Because η is a Gaussian variable, one can take the average over the noise η to see that the replica partition function can be written as (for pt-to-pt case)

$$\langle Z^N(x,t) \rangle = \langle x | e^{-H_N t} | 0 \rangle$$

where H_N is the Hamiltonian of the δ-Bose gas,

$$H_N = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j \neq k}^{N} \delta(x_j - x_k).$$

Remark: More generally, the N point correlation function satisfies

$$\frac{d}{dt}\left\langle \prod_{i=1}^{N} Z(x_i,t) \right\rangle = H_N \left\langle \prod_{i=1}^{N} Z(x_i,t) \right\rangle$$

Remember $h = \log Z$. We are interested not only in the average $\langle h \rangle$ but the full distribution of h. Here we compute the generating function $G_t(s)$ of the replica partition function,

$$G_t(s) = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(0,t) \right\rangle e^{N\frac{\gamma_t^3}{12}}$$

with $\gamma_t = (t/2)^{1/3}$. This turns out to be written as a Fredholm determinant. We apply the inversion formula to recover the p.d.f for h. But for the KPZ, $\langle Z^N \rangle \sim e^{N^3}$.

4. Stationary case

2012-2013 Imamura S

- Narrow wedge is technically the simplest.
- Flat case is a well-studied case in surface growth
- Stationary case is important for stochastic process and nonequilibrium statistical mechanics
  - Two-point correlation function
  - Experiments: Scattering, direct observation
  - A lot of approximate methods (renormalization, mode-coupling, etc.) have been applied to this case.
  - Nonequilibrium steady state(NESS): No principle. Dynamics is even harder.

Modification of initial condition

Two sided BM

$$h(x,0) = \begin{cases} B_-(-x), & x < 0, \\ B_+(x), & x > 0, \end{cases}$$

where $B_\pm(x)$ are two independent standard BMs

We consider a generalized initial condition

$$h(x,0) = \begin{cases} \tilde{B}(-x) + v_- x, & x < 0, \\ B(x) - v_+ x, & x > 0, \end{cases}$$

where $B(x), \tilde{B}(x)$ are independent standard BMs and $v_\pm$ are the strength of the drifts.

Result

For the generalized initial condition with $v_\pm$

$$F_{v_\pm,t}(s) := \mathbf{Prob}\left[h(x,t) + \gamma_t^3/12 \le \gamma_t s\right]$$
$$= \frac{\Gamma(v_+ + v_-)}{\Gamma(v_+ + v_- + \gamma_t^{-1} d/ds)} \left[1 - \int_{-\infty}^{\infty} du e^{-e^{\gamma_t(s-u)}} \nu_{v_\pm,t}(u)\right]$$

Here $\nu_{v_\pm,t}(u)$ is expressed as a difference of two Fredholm determinants,

$$\nu_{v_\pm,t}(u) = \det\left(1 - P_u(B_t^\Gamma - P_{\mathrm{Ai}}^\Gamma)P_u\right) - \det\left(1 - P_u B_t^\Gamma P_u\right),$$

where P_s represents the projection onto (s,∞),

$$P_{\mathrm{Ai}}^\Gamma(\xi_1,\xi_2) = \mathrm{Ai}_\Gamma^\Gamma\left(\xi_1, \frac{1}{\gamma_t}, v_-, v_+\right) \mathrm{Ai}_\Gamma^\Gamma\left(\xi_2, \frac{1}{\gamma_t}, v_+, v_-\right)$$

$$B_t^{\Gamma}(\xi_1, \xi_2) = \int_{-\infty}^{\infty} dy \frac{1}{1 - e^{-\gamma_t y}} \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_1 + y, \frac{1}{\gamma_t}, v_-, v_+\right)$$
$$\times \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_2 + y, \frac{1}{\gamma_t}, v_+, v_-\right),$$

and

$$\mathrm{Ai}_{\Gamma}^{\Gamma}(a, b, c, d) = \frac{1}{2\pi} \int_{\Gamma_{i\frac{d}{b}}} dz e^{iza + i\frac{z^3}{3}} \frac{\Gamma(ibz + d)}{\Gamma(-ibz + c)},$$

where Γ_{z_p} represents the contour from $-\infty$ to ∞ and, along the way, passing below the pole at $z = id/b$.

Height distribution for the stationary KPZ equation

$$F_{0,t}(s) = \frac{1}{\Gamma(1+\gamma_t^{-1}d/ds)} \int_{-\infty}^{\infty} du \gamma_t e^{\gamma_t(s-u)+e^{-\gamma_t(s-u)}} \nu_{0,t}(u)$$

where $\nu_{0,t}(u)$ is obtained from $\nu_{v_\pm,t}(u)$ by taking $v_\pm \to 0$ limit.

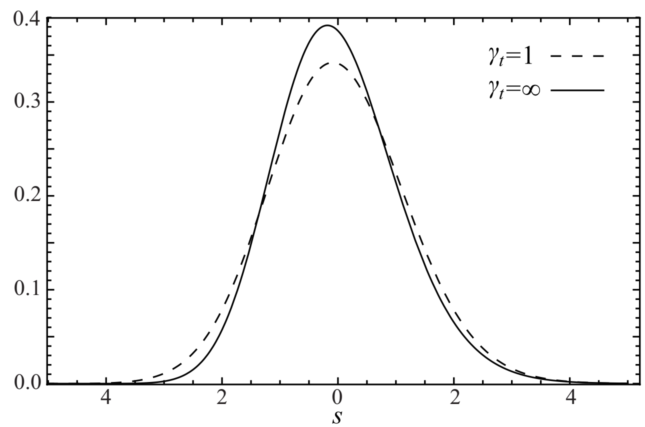


Figure 1: Stationary height distributions for the KPZ equation for $\gamma_t = 1$ case. The solid curve is F_0.

Stationary 2pt correlation function

$$C(x,t) = \langle (h(x,t) - \langle h(x,t) \rangle)^2 \rangle$$

$$g_t(y) = (2t)^{-2/3} C\left((2t)^{2/3} y, t\right)$$

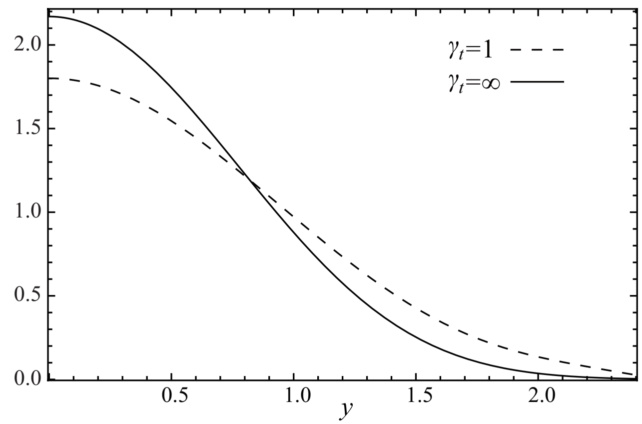


Figure 2: Stationary 2pt correlation function $g_t''(y)$ for $\gamma_t = 1$. The solid curve is the corresponding quantity in the scaling limit $g''(y)$.

5. Further developments

O'Connell

Semi-discrete finite temperature directed polymer $\cdots$ quantum Toda lattice

Partition function

$$Z_t^N(\beta) = \int_{0<t_1<\ldots<t_{N-1}<t} \exp \beta \left(\sum_{i=1}^{N} (B_i(t_i) - B_i(t_{i-1}) \right)$$

$B_i(t)$: independent Brownian motions

Macdonald process

2011 Borodin, Corwin

- Measure written as

$$\frac{1}{Z} P_\lambda(a) Q_\lambda(b)$$

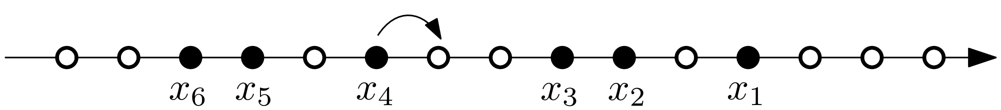
  where P, Q are Macdonald polynomials.

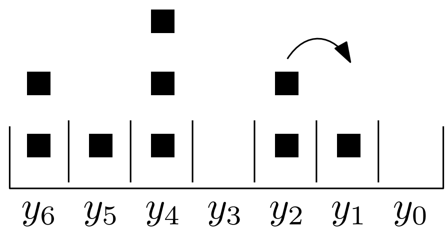
- A generalization of Schur measure
- Includes Toda, Schur and KPZ as special and limiting cases
- Non-determinantal but expectation value of certain "observables" can be written as Fredholm determinants.

q-TASAEP $\cdots$ Rigorous replica

Borodin-Corwin-S

q-TASEP particle i hops with rate $1 - q^{x_{i-1}-x_i-1}$.





The dynamics of the gaps $y_i = x_{i-1} - x_i - 1$ is a version of the zero range process in which a particle hops to the right site with rate $1 - q^{y_i}$. The generator of the process can be written in terms of "q-deformed boson". (1998 Sasamoto, Wadati)

Defining KPZ equation without Cole-Hopf

2011 Hairer

- Universality in the KPZ problems. The Cole-Hopf does not work for most models which are expected to be in the KPZ universality class.
- Rough path and renormalization.
- Coincide with the Cole-Hopf solution.
- Various generalizations to other non-linear SPDE.
- Proving the convergence to the KPZ equation becomes easy.

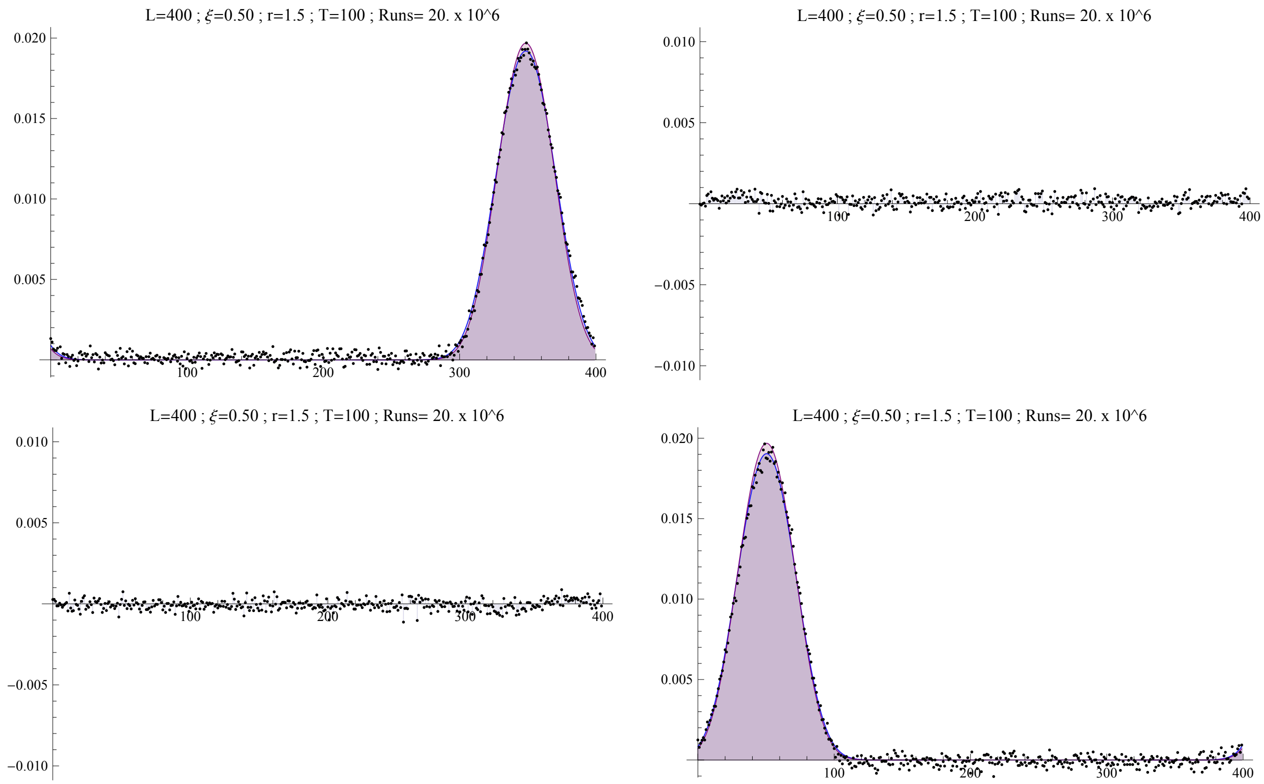
Systems with many conserved quantities

Conjecture 2011- Beijeren, Spohn, etc

For rather generic 1D systems with more than one conserved quantities, the correlation functions for "normal modes" are described by the single component KPZ correlation functions.

- FPU chain, hard-point particles with alternating mass, quantum systems, etc.
- There are three conserved quantities.

KPZ scaling function in MC simulation of multi-species ASEP



2013 Ferrari S Spohn

Simulations in 2D

In higher dimensions, there had been several conjectures for exponents.

There are almost no rigorous results.

2012 Halpin-Healy

New extensive Monte-Carlo simulations on the distributions.

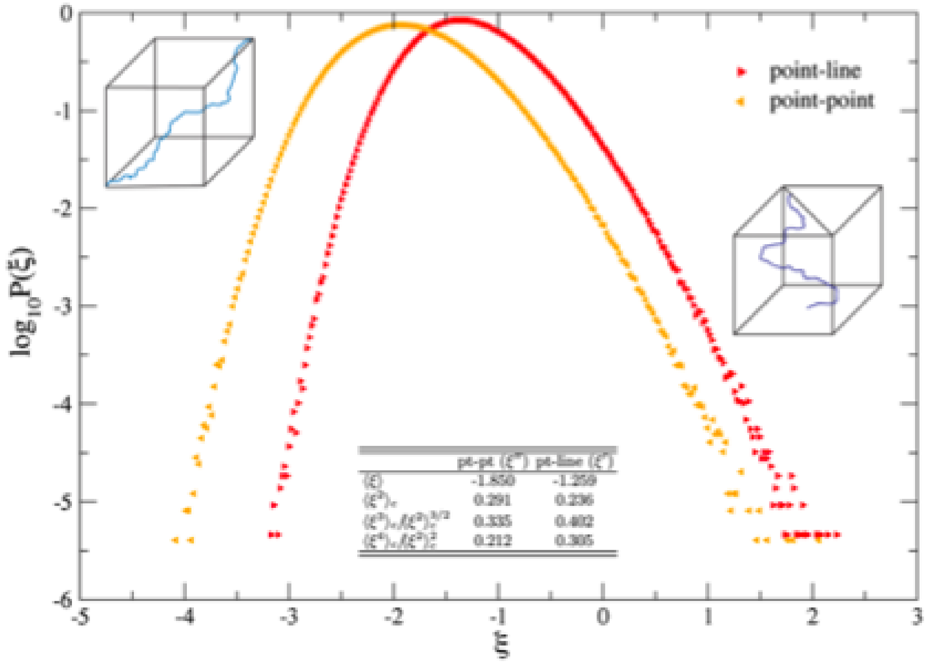


FIG. 4 (color online). Universal PDFs: 2 + 1 DPRM *point-point* and *point-line* geometries. Table inset: Distribution moments.

New universal distributions?

6. Summary

- The KPZ equation is a well-known equation for describing surface growth.
- The KPZ universality may be applicable to wider class of systems than previously thought. Systems with more than one conserved quantities, quantum systems, etc...
- The understanding of the convergence to the KPZ equation is getting better.
- The KPZ universality and the universality of the KPZ equation are different.
- 基研研究会"界面ゆらぎと KPZ 普遍クラスに関する数学・理論・実験的アプローチの融合"8/20-23