Operational characterization of quantum nonlocality

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JST PRESTO

6 Mar. 2019

Motivation

- Quantum physics has a beautiful mathematical representation.
- But, we do not have any "explanation" for the quantum physics.
- We need to find postulates of quantum physics.

  Postulate: Similar to axiom in math. But, it must be testable by experiments, e.g.,

  - Information cannot be transmitted faster than light.
  - A communication complexity is not always equal to 1.

Quantum physics

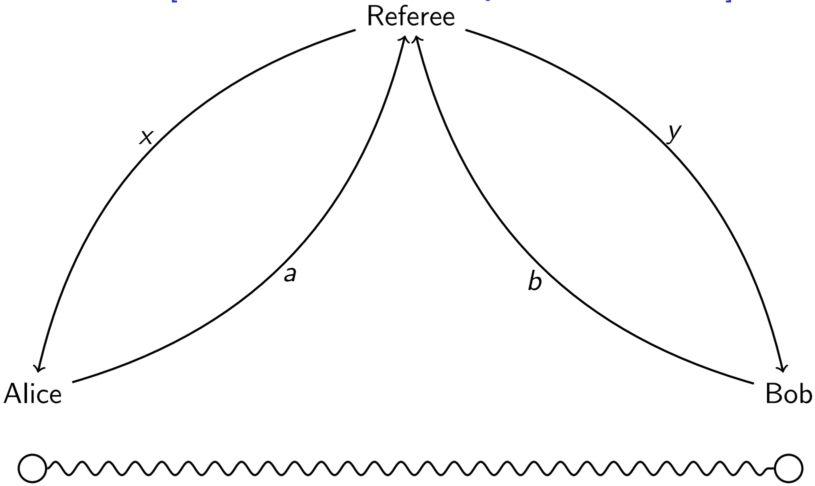
- There is no concept of “quantum probability”.

- A probability is always expressed by a tuple of non-negative values with sum 1.

- There are concepts of “state” and “measurement”.
  - State: Environment.
  - Measurement: Operation to a state for getting an outcome.
  - Probability of an output $a \in \mathcal{A}$ when a measurement $x \in \mathcal{X}$ is chosen is $P(a \mid x)$.

CHSH game [Bell 1964 **12736**]

[Clauser, Horne, Shimony, Holt 1969 **6564**]



Alice and Bob win iff $a \oplus b = x \wedge y$.

CHSH winning probability

- The maximum CHSH winning probability in classical physics is $3/4 = 0.75$.

$$\begin{array}{ccc} b1 & \neq & a1 \\ \| & & \| \\ a0 & = & b0 \end{array}$$

- The maximum CHSH winning probability in quantum physics is $(2+\sqrt{2})/4 \approx 0.854$ [Tsirelson 1980 **1380**].

Locality (Hidden variable model)

Joint preparation and independent measurements.

Probability distribution $P(a, b \mid x, y)$ is said to be **local** if

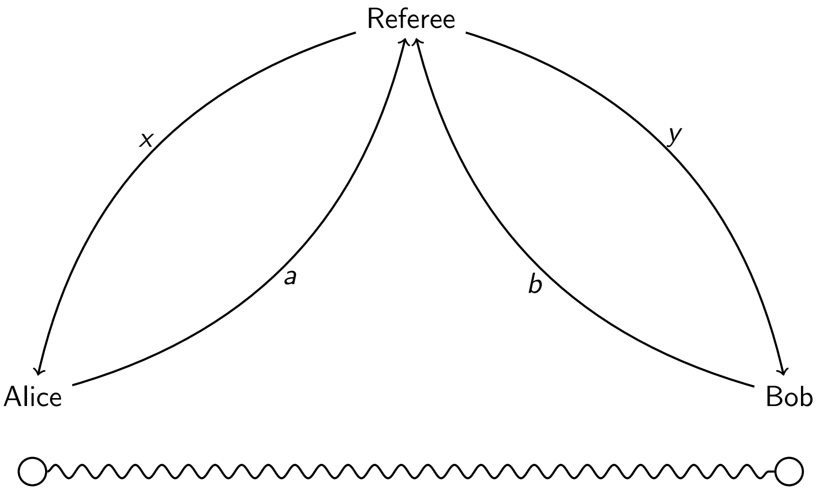
$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda).$$

Equivalently, there exists a joint distribution $P(a_0, a_1, b_0, b_1)$.

Quantum physics allow nonlocal behaviors.

[Einstein, Podolsky, Rosen 1935, **17516**]

Two-party statistics



$$P(a, b \mid x, y), \qquad \forall a, b \in \{0, 1\}, x, y \in \{0, 1\}$$

No-signaling condition

The marginal distribution of a (b) cannot depend on y (x), respectively.

$$\sum_{b\in\{0,1\}} P(a,b \mid x,0) = \sum_{b\in\{0,1\}} P(a,b \mid x,1), \qquad \forall a,\ x \in \{0,1\}$$
$$\sum_{a\in\{0,1\}} P(a,b \mid 0,y) = \sum_{a\in\{0,1\}} P(a,b \mid 1,y), \qquad \forall b,\ y \in \{0,1\}.$$

The 8-dimensional linear space and no-signaling polytope

$$\sum_{a\in\{0,1\},\, b\in\{0,1\}} P(a, b \mid x, y) = 1, \qquad x \in \{0,1\},\ y \in \{0,1\}.$$

$$\sum_{b\in\{0,1\}} P(0, b \mid 0, 0) = \sum_{b\in\{0,1\}} P(0, b \mid 0, 1)$$

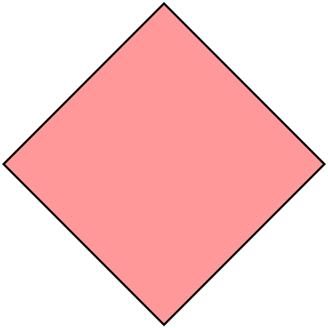
$$\sum_{b\in\{0,1\}} P(0, b \mid 1, 0) = \sum_{b\in\{0,1\}} P(0, b \mid 1, 1)$$

$$\sum_{a\in\{0,1\}} P(a, 0 \mid 0, 0) = \sum_{a\in\{0,1\}} P(a, 0 \mid 1, 0)$$

$$\sum_{a\in\{0,1\}} P(a, 0 \mid 0, 1) = \sum_{a\in\{0,1\}} P(a, 0 \mid 1, 1).$$

$16 - 8 = 8$-dimensional linear space.

No-signaling polytope



Local polytope

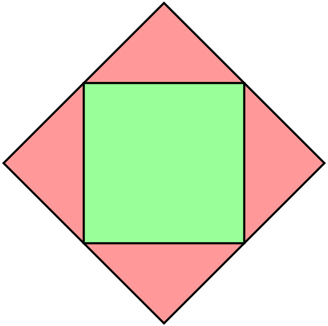
Deterministic choice

$$a = A(x), \qquad\qquad b = B(y).$$

Local polytope

$$\mathrm{conv}\left(\left\{\left\{P(a,b \mid x,y) = \delta_{(a,b),(A(x),B(y))}\right\}_{a,b,x,y} \mid A, B \in \{0,1\}^{\{0,1\}}\right\}\right).$$

No-signaling polytope and local polytope



CHSH inequality: Facets of the local polytope

$$\sum_{a\oplus b = x\wedge y} P(a,b \mid x,y) \le 3, \qquad \sum_{a\oplus b \ne x\wedge y} P(a,b \mid x,y) \le 3$$

$$\sum_{a\oplus b = \overline{x}\wedge y} P(a,b \mid x,y) \le 3, \qquad \sum_{a\oplus b \ne \overline{x}\wedge y} P(a,b \mid x,y) \le 3$$

$$\sum_{a\oplus b = x\wedge \overline{y}} P(a,b \mid x,y) \le 3, \qquad \sum_{a\oplus b \ne x\wedge \overline{y}} P(a,b \mid x,y) \le 3$$

$$\sum_{a\oplus b = \overline{x}\wedge \overline{y}} P(a,b \mid x,y) \le 3, \qquad \sum_{a\oplus b \ne \overline{x}\wedge \overline{y}} P(a,b \mid x,y) \le 3$$

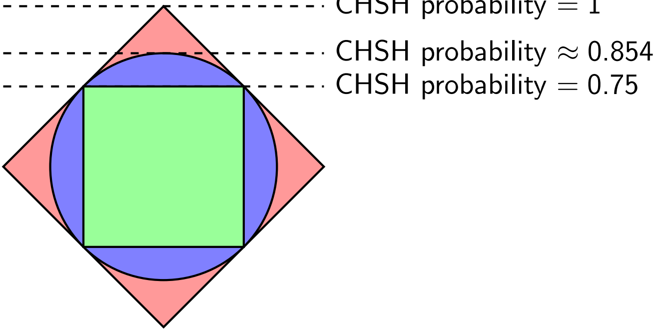
CHSH inequality [Clauser, Horne, Shimony, Holt 1969 **6564**].
CHSH inequality is the only non-trivial facets [Froissard 1981 **111**], [Fine 1982 **991**].

No-signaling condition admits CHSH probability 1

$$
\begin{aligned}
P(0,0 \mid 0,0) &= P(1,1 \mid 0,0) = 1/2 \\
P(0,0 \mid 0,1) &= P(1,1 \mid 0,1) = 1/2 \\
P(0,0 \mid 1,0) &= P(1,1 \mid 1,0) = 1/2 \\
P(0,1 \mid 1,1) &= P(1,0 \mid 1,1) = 1/2
\end{aligned}
$$

[Popescu and Rohrlich 1994 **1122**]

No-signaling polytope, local polytope and quantum correlation



Question:

Why does quantum physics prohibits CHSH probability greater than $(2+\sqrt{2})/4 \approx 0.854$?

Topics

- $p_{\mathrm{CHSH}} = 1 \implies$ Communication complexity (CC) of arbitrary function is 1 bit.
  [van Dam 2013 (quant-ph/0501159) (Ph.D. thesis 1999) **106**]

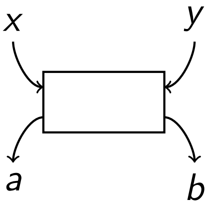
- $p_{\mathrm{CHSH}} > (3 + \sqrt{6})/6 \approx 0.908 \implies$ CC of arbitrary function is 1 bit.
  [Brassard, Buhrman, Linden, Méthot, Tapp, Unger 2006 **291**]

- $p_{\mathrm{CHSH}} > (2 + \sqrt{2})/4 \approx 0.854 \implies$ Information causality is violated.
  [Pawłowki, Paterek, Kaszlikowski, Scarani, Winter, Zukowki 2009 **462**]

- Brassard et al.'s result cannot be improved by generalizations of their techniques [Mori 2016].

Nonlocal box

Abstract device with two input ports and two output ports.



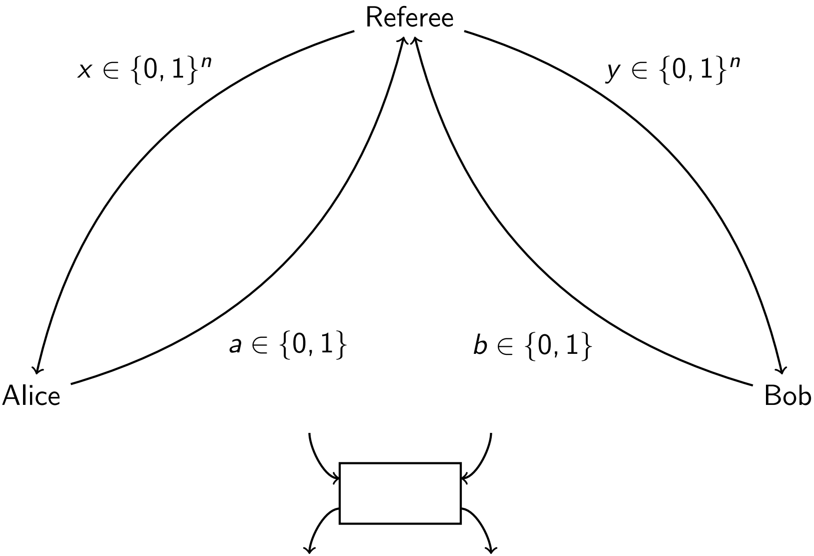
Isotropic nonlocal box

$$P(a, b \mid x, y) = \begin{cases} \frac{p_{\mathrm{CHSH}}}{2}, & \text{if } a \oplus b = x \wedge y \\ \frac{1-p_{\mathrm{CHSH}}}{2}, & \text{if } a \oplus b \neq x \wedge y. \end{cases}$$

This does not lose generality since

$$\begin{aligned} x \wedge y &= (x \oplus r_1) \wedge (y \oplus r_2) \oplus x \wedge r_2 \oplus r_1 \wedge y \oplus r_1 \wedge r_2 \\ &= a \oplus b \oplus e \oplus x \wedge r_2 \oplus r_1 \wedge y \oplus r_1 \wedge r_2 \\ &= (a \oplus x \wedge r_2 \oplus r_1 \wedge r_2) \oplus (b \oplus r_1 \wedge y) \oplus e \end{aligned}$$

XOR game



Alice and Bob win iff $a \oplus b = f(x, y)$.

PR box gives a winning probability 1

[van Dam 2013 (arXiv 2005) (PhD. thesis 1999) **168**]

If the CHSH probability is 1, a winning probability of any XOR game is 1 !

Any boolean function can be represented by a $\mathbb{F}_2$-polynomial.

$$f(x,y) = \bigoplus_i A_i(x) \wedge B_i(y).$$

Recall Alice and Bob have nonlocal boxes with

$$\Pr(a \oplus b = x \wedge y) = 1$$

for any $(x,y) \in \{0,1\}^2$,

$$\begin{aligned} \bigoplus_i A_i(x) \wedge B_i(y) &= \bigoplus_i (a_i \oplus b_i) \\ &= \left(\bigoplus_i a_i \right) \oplus \left(\bigoplus_i b_i \right). \end{aligned}$$

Bias

For a probability $p \in [1/2, 1]$, $\beta := 2p - 1 \in [0, 1]$ is called a bias.
In other word,

$$p = \frac{1 + \beta}{2}.$$

Let β be a bias of the CHSH probability p_{CHSH}.

- $p_{\mathrm{CHSH}} = 3/4 \iff \beta = 1/2$.
- $p_{\mathrm{CHSH}} = (2 + \sqrt{2})/4 \iff \beta = 1/\sqrt{2}$.
- $p_{\mathrm{CHSH}} = 1 \iff \beta = 1$.

- If X is ± 1 random variable, the bias (for a prob. of 1) is $\mathbb{E}[X] = \frac{1+\beta}{2} - \frac{1-\beta}{2} = \beta$.
- If X and Y are independent 0-1 random variables with bias (for a prob. of 0) β_X and β_Y, respectively, the bias of $X \oplus Y$ is $\beta_X \beta_Y$.

Constant winning probability

[Brassard, Buhrman, Linden, Méthot, Tapp, Unger 2006 **291**]

$p_{\text{CHSH}} > \frac{3+\sqrt{6}}{6} \approx 0.908 \iff \beta > \sqrt{\frac{2}{3}}$
$\implies$ A winning probability of any XOR game is constant $(> \frac{1}{2})$.

By using shared random bits $r \in \{0,1\}^n$ and Bob's private random bit $r' \in \{0,1\}$,

$$a = f(x, r)$$
$$b = \begin{cases} 0, & \text{if } y = r \\ r', & \text{otherwise.} \end{cases}$$

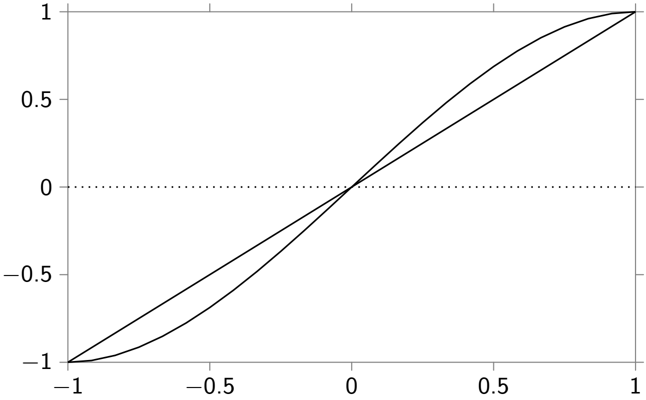
$a \oplus b = f(x, y)$ with probability

$$\frac{1}{2^n} + \left(1 - \frac{1}{2^n}\right)\frac{1}{2} = \frac{1+2^{-n}}{2}.$$

Bias amplification by Maj_3

$$\mathrm{Maj}_3(z_1, z_2, z_3) = \frac{1}{2}\left(z_1 + z_2 + z_3 - z_1 z_2 z_3\right)$$

$$\mathbb{E}\left[\mathrm{Maj}_3(z_1, z_2, z_3)\right] = \frac{3}{2}\epsilon - \frac{1}{2}\epsilon^3$$

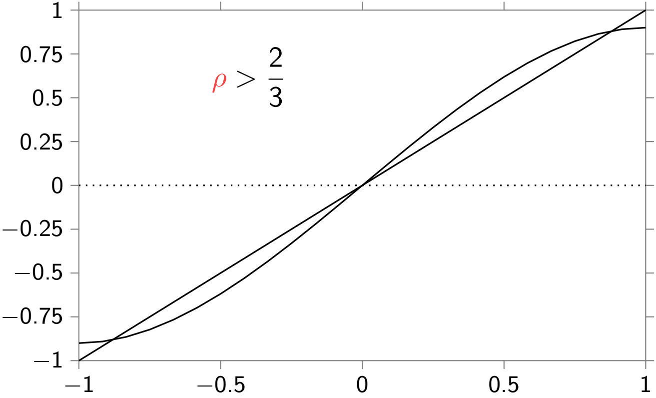


Bias amplification by noisy Maj_3

[von Neumann 1956 **2708**]

$$\mathrm{Maj}_3(z_1, z_2, z_3) = \frac{1}{2}(z_1 + z_2 + z_3 - z_1 z_2 z_3)$$

$$\mathbb{E}\left[y \mathrm{Maj}_3(z_1, z_2, z_3)\right] = \rho\left(\frac{3}{2}\epsilon - \frac{1}{2}\epsilon^3\right)$$



Probability of succeeding of computation of Maj_3

$$\mathrm{Maj}_3(z_1, z_2, z_3) = z_1 z_2 \oplus z_2 z_3 \oplus z_3 z_1$$

$$\begin{aligned}
&\mathrm{Maj}_3(a_1 \oplus b_1, a_2 \oplus b_2, a_3 \oplus b_3)\\
&= (a_1 \oplus b_1)(a_2 \oplus b_2) \oplus (a_2 \oplus b_2)(a_3 \oplus b_3) \oplus (a_3 \oplus b_3)(a_1 \oplus b_1)\\
&= (a_1 \oplus a_2)(b_2 \oplus b_3) \oplus (a_2 \oplus a_3)(b_1 \oplus b_2)\\
&\qquad \oplus a_1 a_2 \oplus a_2 a_3 \oplus a_3 a_1\\
&\qquad \oplus b_1 b_2 \oplus b_2 b_3 \oplus b_3 b_1\\
&= (\alpha_0 \oplus \beta_0 \oplus e_0) \oplus (\alpha_1 \oplus \beta_1 \oplus e_1)\\
&\qquad \oplus a_1 a_2 \oplus a_2 a_3 \oplus a_3 a_1\\
&\qquad \oplus b_1 b_2 \oplus b_2 b_3 \oplus b_3 b_1\\
&= (\alpha_0 \oplus \alpha_1 \oplus a_1 a_2 \oplus a_2 a_3 \oplus a_3 a_1) \oplus (\beta_0 \oplus \beta_1 \oplus b_1 b_2 \oplus b_2 b_3 \oplus b_3 b_1) \oplus e_0 \oplus e_1.
\end{aligned}$$

$$\beta^2 > \frac{2}{3} \iff \beta > \sqrt{\frac{2}{3}} \iff p > \frac{1 + \sqrt{\frac{2}{3}}}{2} = \frac{3 + \sqrt{6}}{6} \approx 0.908.$$

Generalization of Brassard et al's protocol

- Why Maj_3 ?

- Replace Maj_3 with arbitrary boolean function.

- Two important parameters:
  - 2: Number of nonlocal boxes for the computation.
  - 2/3: Threshold for the bias amplification.

- We showed that the Maj_3 is the unique optimal function in a simple generalization [Mori, Phys. Rev. A 94, 052130, 2016].

Information causality

[Pawłowki, Paterek, Kaszlikowski, Scarani, Winter, Zukowki 2009 **462**]

Information causality:

If Alice communicates m bits to Bob, the total information obtainable by Bob cannot be greater than m.

Alice has 2^n bits. Bob wants to know one of Alice's 2^n bits. Alice doesn't know which bit Bob wants to know.

IC says that Alice has to send 2^n bits.

Above the quantum limit 0.854, Alice only has to send 1.99^n bits.

Address function

$$\mathrm{Addr}_n(x_0, \dots, x_{2^n-1}, y_1, \dots, y_n) := x_y$$

where $y := \sum_{i=1}^{n} y_i 2^{i-1}$.

Theorem ([Pawłowski, Paterek, Kaszlikowski, Scarani, Winter, Zukowski 2009 **462**])

There is an adaptive protocol of the XOR game for the address function with bias β^n.

Proof

Induction.

For $n = 1$, from

$$\mathrm{Addr}_1(x_0, x_1, y_1) = x_0 \oplus y_1(x_0 \oplus x_1)$$

there is a non-adaptive protocol with bias β.

Proof (Cont'd).

$$\mathrm{Addr}_n(x_0, \dots, x_{2^n-1}, y_1, \dots, y_n) = \mathrm{Addr}_1(z_0, z_1, y_n)$$

where

$$z_0 := \mathrm{Addr}_{n-1}(x_0, \dots, x_{2^{n-1}-1}, y_1, \dots, y_{n-1})$$
$$z_1 := \mathrm{Addr}_{n-1}(x_{2^{n-1}}, \dots, x_{2^n-1}, y_1, \dots, y_{n-1}).$$

$$\begin{aligned}
&\mathrm{Addr}_1(z_0, z_1, y_n) = \mathrm{Addr}_1(a_0 \oplus b_0 \oplus e_0, a_1 \oplus b_1 \oplus e_1, y_n) \\
&= \mathrm{Addr}_1(a_0, a_1, y_n) \oplus b_{y_n} \oplus e_{y_n} \\
&= a' \oplus b' \oplus e' \oplus b_{y_n} \oplus e_{y_n} \\
&= a' \oplus (b' \oplus b_{y_n}) \oplus (e' \oplus e_{y_n}).
\end{aligned}$$

This protocol has bias β^n. □

Repetition

The 1 bit communication has error probability $\epsilon := \frac{1-\beta^n}{2}$.
The m bits communication has error probability $\leq \left(2\sqrt{\epsilon(1-\epsilon)}\right)^m$.

From

$$\left(2\sqrt{\epsilon(1-\epsilon)}\right)^m = (1-\beta^{2n})^{\frac{m}{2}}$$

error probability goes to zero if

$$m \gg \beta^{-2n}.$$

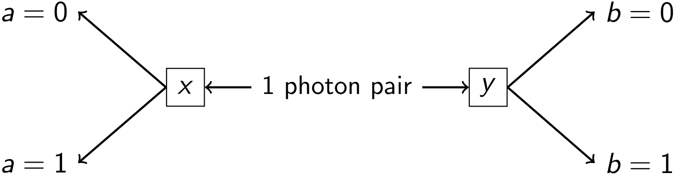
If $\beta > 1/\sqrt{2}$, then $\beta^{-2} < 2$.

If CHSH probability is greater than the quantum limit,

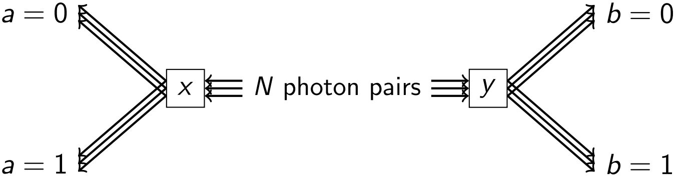
1.99^n bits communication allows Bob to select arbitrary one bit from Alice's 2^n bits.

Macroscopic locality

Nature should not exhibit nonlocal behaviour in macroscopic setting.



Microscopic experiment of nonlocality.



Macroscopic experiment of nonlocality (with precision $O(\sqrt{N})$).
[Navascués, Wunderlich, 2009, **219**]

Central limit theorem

For fixed x and y,
$\{(N(a;x) - \mathbb{E}[N(a;x)])/\sqrt{N}, (N(b;y) - \mathbb{E}[N(b;y)])/\sqrt{N}\}_{a,b}$
weakly converges to the normal distribution. Assume

$$\mathbb{E}[a_x] = 0, \qquad \mathbb{E}[b_y] = 0, \qquad \mathbb{E}[a_x b_y] = (-1)^{x \wedge y}\beta.$$

Then, the nonlocal box is macroscopically local if and only if $\exists\lambda \in [-1, +1]$ such that

$$\Gamma(\lambda) := \begin{bmatrix} 1 & \lambda & \beta & \beta \\ \lambda & 1 & \beta & -\beta \\ \beta & \beta & 1 & \lambda \\ \beta & -\beta & \lambda & 1 \end{bmatrix} \succeq 0$$

This condition shows $\beta \leq \frac{1}{\sqrt{2}}$.

Toward characterizing the quantum correlation

Theorem ([Navascués and Wunderlich 2009, **219**])

Quantum physics satisfies the macroscopic locality.

Theorem ([Navascués and Wunderlich 2009, **219**])

There exists a macrospically local distribution with biased marginals attaining the Tsirelson bound. Hence, the macroscopic locality alone cannot characterize the quantum correlation.

Theorem

Macroscopic locality completely characterizes the bipartite quantum correlation with binary outputs with unbiased marginals.