Sachdev-Ye-Kitaev type models with dissipations

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1 Opening

今日のトピック： 多体開放量子系における強い相互作用と散逸の効果

Based on these recent papers:

- "*Lindbladian dynamics of the Sachdev-Ye-Kitaev model*" [Kulkarni, Numasawa, SR, arXiv:2112.13489]
- "*Dynamical quantum phase transitions in SYK Lindbladians*" [Kawabata, Kulkarni, Li, Numasawa, SR, arXiv:2210.04093]
- "*Symmetry of open quantum systems: Classification of dissipative quantum chaos*" [Kawabata, Kulkarni, Li, Numasawa, SR, arXiv:2212.00605]

These works cannot be completed without these wonderful collaborators:

Anish Kulkarni (Princeton), Jiachen Li (Princeton),
川畑幸平(Princeton), 沼澤宙郎（物性研）

Special thanks also go to Lucas Sa (U Lisbon) who also worked on very closely related papers. And we discussed quite a lot.

Outline

- 背景
- 多体リンドブラディアンの対称性クラス
- SYK リンドブラディアンの振る舞い （特に、dissipative form factor)

2 背景

- 孤立量子系：ユニタリ時間発展
  - E.g., トポロジカルな時間発展、多体系の量子カオス（ETHや量子情報の撹拌）

- 開放量子系（i.e.,環境との相互作用）：量子チャンネル
  - E.g., エンタングルメント転移 in monitored circuit;
    Preparation of desired (topological) states by dissipative dynamics, etc.
  - Unique properties not seen in unitary dynamics?

- 本日の話題：多体開放量子系の「ランダム行列理論」
- トイ模型（SYK型の模型）を中心に強い相互作用と散逸のinterplayを議論する。

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3 ランダム行列理論、多体の量子カオス

- 複雑な原子核、メゾスコピック系、数論、etc.
- 非可積分系の様々な普遍的な性質、e.g., 準位統計
- 量子多体系、統計力学への応用、e.g., ETH, Page curve

E.g.:
拡張ハバード模型の準位間隔分布 [Rigol-Santos (10)]

Kicked Ising model のスペクトルフォームファクター: $K(t) = |\mathrm{Tr}\, e^{-iHt}|^2$
[Prosen (18)]

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4 SYK 模型

- q-body SYK model:

$$H_{\mathrm{SYK}} = i^{q/2} \sum_{i_1 < \cdots < i_q} J_{i_1 \cdots i_q} \psi_{i_1} \cdots \psi_{i_q}$$

$$\{\psi_i, \psi_j\} = \delta_{ij}, \quad i, j = 1, \cdots, N$$

$$J_{i_1 \cdots i_q} = \mathrm{random}$$

- ラージN極限で可解(N= フェルミオンの数)
- "Maximally chaotic". リヤプノフ指数のMaldacena-Shenker-Stanford bound (16)を飽和.
- 2次元重力理論（JT重力）による有効的な記述
  [Maldacena-Stanford, 16] [Maldacena-Stanford-Yang, 16]
- 非フェルミ流体などへの応用
- 多体開放量子系では？

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5 Lindbladian Dynamics

- 量子チャンネルに対する時間発展の生成子（マルコフ性を仮定） [Lindblad, 76, Gorini-Kossakowski-Sudarshan, 76]

$$
\begin{aligned}
&\frac{d\rho}{dt} = \mathbb{L}(\rho), \\
&\mathbb{L}(\rho) = -i[H, \rho] + \sum_{\alpha} \Big[L_\alpha \rho L_\alpha^\dagger - \frac{1}{2} \{ L_\alpha^\dagger L_\alpha, \rho \} \Big].
\end{aligned} \tag{1}
$$

- 第一項：ユニタリ時間発展。
- 第二項：非ユニタリ；環境との相互作用 "dissipators" or "jump operators".
- トレース保存: $\mathrm{Tr}\,\rho(t) = 1$.
- 例：二準位系: $H = \omega \sigma_z$ and $L \propto \sigma_-$, $L \propto \sigma_z$

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6 SYK Lindbladians

- Model (i); non-random, linear jump operators

$$
L^i = \sqrt{\mu} \psi^i, \quad i = 1, \cdots, N.
$$

- Model (ii); Random, quadratic jump operators

$$
\begin{aligned}
&L^a = \sum_{1 \le i < j \le N} K_{ij}^a \psi_i \psi_j, \quad K_{ij}^a \in \mathbb{C}, \quad a = 1, \cdots, M, \\
&\left\langle K_{ij}^a \right\rangle = 0, \quad \left\langle \left| K_{ij}^a \right|^2 \right\rangle = \frac{K^2}{N^2} \quad \forall i, j, a \quad \text{(no sum)}.\ R := \frac{M}{N}
\end{aligned}
$$

- More generally, we can consider p-body jump operators

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7 Doubled Hilbert space

- 密度行列を二重化ヒルベルト空間のベクトルと思うと便利("vectorization")

$$
\begin{aligned}
&|i\rangle\langle j| \to |i\rangle |j\rangle^* \\
&\rho \to |\rho\rangle\rangle \in \mathcal{H} \otimes \mathcal{H}^*, \\
&\mathbb{L} \to \mathcal{L} = \text{non-hermitian operator acting on } \mathcal{H} \otimes \mathcal{H}^*
\end{aligned} \tag{2}
$$

- Schwinger-Keldysh, 熱場のダイナミクス[高橋・梅沢(75)]
- Choi-Jamiolkowski isomorphism, 富田・竹崎理論
- トレースは二重化ヒルベルト空間のトレースに対応する。 $\mathrm{Tr}_{\mathcal{H}}\,\rho(t) = \langle\langle I | \rho(t) \rangle\rangle$.
- トレースの巡回性は、反ユニタリなモジュラー共役演算子 $\mathcal{J}$をつかって表される

$$
\mathcal{J}(A \otimes B)\mathcal{J}^{-1} = B \otimes A, \quad \mathcal{J} z \mathcal{J}^{-1} = z^*
$$

- Can draw an analogy with unitary dynamics (well, sort of).
- $\mathcal{L}$は二重ヒルベルト空間における非エルミート演算子（ただしいくつかの制約がある）

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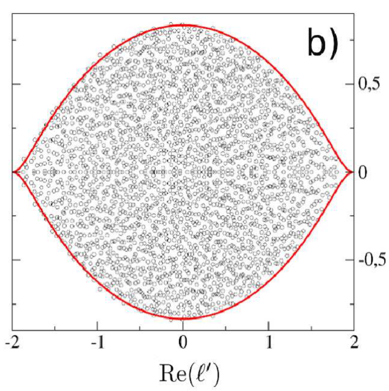
8 リンドブラディアンに対するランダム行列理論

- Bohigas-Giannoni-Schmit conjecture?
- 量子情報の撹拌、operator growth, ...
- ...

リンドブラディアンに対するランダム行列理論：

- S. Denisov, T. Laptyeva, W. Tarnowski, D. Chru´sci´nski, and K. Zyczkowski,
Universal Spectra of Random Lindblad Operators, Phys. Rev. Lett. 123, 140403 (2019);
- T. Can, V. Oganesyan, D. Orgad, and S. Gopalakrishnan,
Spectral Gaps and Midgap States in Random Quantum Master Equations,
Phys. Rev. Lett. 123, 234103 (2019);
- K. Wang, F. Piazza, and D. J. Luitz,
Hierarchy of Relaxation Timescales in Local Random Liouvillians,
Phys.Rev. Lett. 124, 100604 (2020);
- T. Can,
Random Lindblad dynamics, Journal of Physics A Mathematical General 52, 485302 (2019);
- L. S´a, P. Ribeiro, and T. Prosen,
Spectral and steady state properties of random Liouvillians, Journal of Physics A Mathematical General 53, 305303 (2020);
- L. S´a, P. Ribeiro, T. Can, and T. Prosen,
Spectral transitions and universal steady states in random Kraus maps and circuits, Phys. Rev. B 102, 134310 (2020)]

スペクトル分布（のサポート）：



Outline

We will discuss three aspects:

- Symmetries
- Dynamical properties
- Stationary properties

Papers:

- "Symmetry of open quantum systems: Classification of dissipative quantum chaos", Kawabata, Kulkarni, Li, Numasawa, SR, arXiv:2212.00605
- "Dynamical quantum phase transitions in SYK Lindbladians", Kawabata, Kulkarni, Li, Numasawa, SR, arXiv:2210.04093

- "Lindbladian dynamics of the Sachdev-Ye-Kitaev model", by Kulkarni, Numasawa, SR, arXiv:2112.13489

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9 対称性 --ユニタリ系の場合

- ユニタリ時間発展(エルミートハミルトニアン)：
  - 3-fold symmetry classification [Wigner-Dyson]
    - 時間反転対称性の有無とタイプ

$$\text{TRS:} \quad \mathcal{T} H \mathcal{T}^{-1} = H \tag{3}$$

    - Unitary (A), Orthogonal (AI), Symplectic (AII)

- 10-fold symmetry classification [Altland-Zirnbauer, AZ]
  - 時間反転対称性(TRS)、粒子正孔対称性(PHS)の有無とタイプ

$$\text{TRS:} \quad \mathcal{T} H \mathcal{T}^{-1} = H, \tag{4}$$
$$\text{PHS:} \quad \mathcal{C} H^{\dagger} \mathcal{C}^{-1} = -H \tag{5}$$
$$\text{CS:} \quad \mathcal{S} H \mathcal{S}^{-1} = -H \tag{6}$$

  - Free quadratic Hamiltonian, interacting Hamiltonians
  - 超伝導体、量子色力学
  - ランダム行列理論、アンダーソン局在、トポロジカル系
  -

Symmetry class		TRS	PHS	TRS†	PHS†	CS
Complex AZ	A	0	0	0	0	0
	AIII	0	0	0	0	1
Real AZ	AI	+1	0	0	0	0
	BDI	+1	+1	0	0	1
	D	0	+1	0	0	0
	DIII	−1	+1	0	0	1
	AII	−1	0	0	0	0
	CII	−1	−1	0	0	1
	C	0	−1	0	0	0
	CI	+1	−1	0	0	1

- 普遍的な順位統計

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10 対称性 --非ユニタリ系の場合

- 非ユニタリ系: 38-fold classification
  [Bernard-Le Clair (01);Kawabata et al (19);Zhou-Lee (19)]

非ユニタリハミルトニアンの対称性の分類

$$\text{TRS} \quad \mathcal{T} H \mathcal{T}^{-1} = H \tag{7}$$
$$\text{PHS}^\dagger \quad \mathcal{C} H \mathcal{C}^{-1} = -H \tag{8}$$
$$\text{SLS} \quad \mathcal{S} H \mathcal{S}^{-1} = -H \tag{9}$$
$$\text{TRS}^\dagger \quad \mathcal{T} H^\dagger \mathcal{T}^{-1} = H \tag{10}$$
$$\text{PHS} \quad \mathcal{C} H^\dagger \mathcal{C}^{-1} = -H \tag{11}$$
$$\text{CS} \quad \Gamma H^\dagger \Gamma^{-1} = -H \tag{12}$$

Symmetry class		TRS	PHS	TRS†	PHS†	CS
Complex AZ	A	0	0	0	0	0
	AIII	0	0	0	0	1
Real AZ	AI	$+1$	0	0	0	0
	BDI	$+1$	$+1$	0	0	1
	D	0	$+1$	0	0	0
	DIII	-1	$+1$	0	0	1
	AII	-1	0	0	0	0
	CII	-1	-1	0	0	1
	C	0	-1	0	0	0
	CI	$+1$	-1	0	0	1
Real AZ†	AI†	0	0	$+1$	0	0
	BDI†	0	0	$+1$	$+1$	1
	D†	0	0	0	$+1$	0
	DIII†	0	0	-1	$+1$	1
	AII†	0	0	-1	0	0
	CII†	0	0	-1	-1	1
	C†	0	0	0	-1	0
	CI†	0	0	$+1$	-1	1

s	AZ class	$t=0$	$t=1$
0	A		$\mathcal{S}$
1	AIII	$\mathcal{S}_+$	$\mathcal{S}_-$

s	AZ class	$t=0$	$t=1$	$t=2$	$t=3$
0	AI		$\mathcal{S}_-$		$\mathcal{S}_+$
1	BDI	$\mathcal{S}_{++}$	$\mathcal{S}_{-+}$	$\mathcal{S}_{--}$	$\mathcal{S}_{+-}$
2	D		$\mathcal{S}_+$		$\mathcal{S}_-$
3	DIII	$\mathcal{S}_{--}$	$\mathcal{S}_{-+}$	$\mathcal{S}_{++}$	$\mathcal{S}_{+-}$
4	AII		$\mathcal{S}_-$		$\mathcal{S}_+$
5	CII	$\mathcal{S}_{++}$	$\mathcal{S}_{-+}$	$\mathcal{S}_{--}$	$\mathcal{S}_{+-}$
6	C		$\mathcal{S}_+$		$\mathcal{S}_-$
7	CI	$\mathcal{S}_{--}$	$\mathcal{S}_{-+}$	$\mathcal{S}_{++}$	$\mathcal{S}_{+-}$

11 Constraints on Linbladians

Lindbladian: free quadratic Lindbladians. [Lieu-McGinley-Cooper (20)]
[Altland-Fleischhauer-Diehl (21), Sa-Ribeiro-Prosen (22), Kawabata-Kulkarni-Li-Numasawa-SR (22)]

- 二重ヒルベルト空間
- New "conditions": E.g., トレースとエルミート性の保存: $\mathrm{Tr}\,\rho(t)=1$, $\rho^\dagger(t)=\rho(t)$.
- エルミート性 $=$ モジュラー共役対称性

$$\mathcal{J}(A \otimes B)\mathcal{J}^{-1} = B \otimes A, \quad \mathcal{J} z \mathcal{J}^{-1} = z^*$$

- $\mathrm{Im}\,\mathrm{Spec}\,\mathcal{L} \leq 0$: 「シフト」が必要 [Kawasaki-Mochizuki-Obuse (22), Prosen (12)]

$$\tilde{\mathcal{L}} := \mathcal{L} - \frac{\mathrm{tr}\,\mathcal{L}}{\mathrm{tr}\,I} I$$

- ユニタリ対称性：Strong v.s. weak

Need to reconsider symmetry analysis with these constraints

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12 Some examples (bosonic)

Dephasing XYZ model with magnetic field:

$$H = -\sum_{i,j}\left(J_{ij}^x X_i X_j + J_{ij}^y Y_i Y_j + J_{ij}^z Z_i Z_j\right) - \sum_i h_i^x X_i$$

$$L_i = \sqrt{\gamma_i} Z_i$$

- Modular conjugation: $\mathcal{J}\mathcal{L}\mathcal{J}^{-1} = \mathcal{J}$ (symmetry about real axis)
- Time-reversal †: $\mathcal{T}\mathcal{L}^\dagger\mathcal{T}^{-1} = \mathcal{L}$
- Particle-hole †: $\mathcal{C}(\mathcal{L} + \sum_i \gamma_i)\mathcal{C}^{-1} = -(\mathcal{L} + \sum_i \gamma_i)$ (symmetry about shifted imaginary axis)
- Weak $\mathbb{Z}_2$ spin flip

$\Longrightarrow$ Symmetry class "BDI †+ $\mathcal{S}_\pm$":

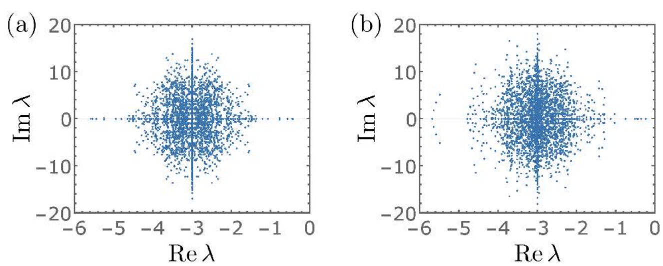

FIG. 1. Dissipative XYZ model in one dimension with periodic boundaries and nearest-neighbor coupling ($L = 6$, $J_{ij}^x = 1.0$, $J_{ij}^y = 0.7$, $J_{ij}^z = 0.9$, $\gamma_i = 0.5$). (a) Complex spectrum in the presence of a magnetic field $h_i^x = 0.2$. (b) Complex spectrum in the presence of magnetic fields $h_i^x = 0.2$, $h_i^z = 0.4$.

- (Right panel) Adding h^z breaks PH † symmetry (lose symmetry about shifted imaginary axis)
- More complete analysis for bosonic case [Sa-Ribeiro-Prosen (22)]

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13 フェルミオン系の場合

$$H_{\mathrm{SYK}} = i^{q/2} \sum_{i_1 < \cdots < i_q} J_{i_1 \cdots i_q} \psi_{i_1} \cdots \psi_{i_q}$$

$$L_m = \sum_{i_1 < \cdots < i_p} K_{m, i_1, \cdots, i_p} \psi_{i_1} \cdots \psi_{i_p}$$

- ブラ空間とケット空間それぞれに作用するフェルミオン演算子、$\{\psi_n^+\}$と $\{\psi_n^-\}$を考える
- 必ず考える対称性：

$$\begin{aligned} \text{Modular conjugation} \quad & \mathcal{J}\psi_n^+\mathcal{J}^{-1} = \psi_n^-, \mathcal{J}\psi_n^-\mathcal{J}^{-1} = \psi_n^+, \mathcal{J}z\mathcal{J}^{-1} = z^* \\ \text{Weak fermion parity} \quad & (-1)^{\mathcal{F}}\psi_n^{\pm}(-1)^{\mathcal{F}} = -\psi_n^{\pm} \end{aligned} \tag{13}$$

- その他の対称性。例：

$$\begin{aligned} \text{Strong fermion parity} \quad & (-1)^{F^{\pm}}\psi_n^{\pm}(-1)^{F^{\pm}} = -\psi_n^{\pm} \quad (-1)^{F^{\pm}}\psi_n^{\mp}(-1)^{F^{\pm}} = \psi_n^{\mp}. \\ \text{Additional symmetry} \quad & \mathcal{R}\psi_n^{\pm}\mathcal{R}^{-1} = \psi_n^{\pm}, \quad \mathcal{R}z\mathcal{R}^{-1} = z^* \end{aligned} \tag{14}$$

- What kinds of **many-body** symmetry classes appear?

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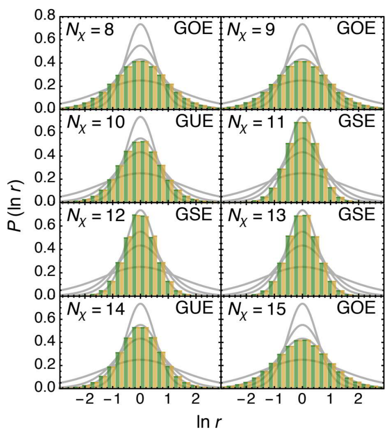
14 SYK模型の対称性

[You-Xu-Ludwig 16]

$$H_{\mathrm{SYK}} = i^{q/2} \sum_{i_1 < \cdots < i_q} J_{i_1 \cdots i_q} \psi_{i_1} \cdots \psi_{i_q}$$

- $(-1)^F$：フェルミオンパリティ保存と $\mathcal{T}$：時間反転対称性を課す
- Projective symmetry analysis: Nに関して８の周期を持つ
  c.f. トポロジカル超伝導体の $\mathbb{Z}_8$分類[Fidkowski-Kitaev]

$N_\chi \pmod 8$	0	1	2	3	4	5	6	7
qdim	1	$\sqrt{2}$	2	$2\sqrt{2}$	2	$2\sqrt{2}$	2	$\sqrt{2}$
lev. stat.	GOE	GOE	GUE	GSE	GSE	GSE	GUE	GOE



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15 Fermionic examples (SYK)

$$\begin{aligned} \text{Modular conjugation} \quad & \mathcal{J}\psi_n^+\mathcal{J}^{-1} = \psi_n^-, \mathcal{J}\psi_n^-\mathcal{J}^{-1} = \psi_n^+, \mathcal{J}z\mathcal{J}^{-1} = z^* \\ \text{Fermion parity} \quad & (-1)^{\mathcal{F}}\psi_n^{\pm}(-1)^{\mathcal{F}} = -\psi_n^{\pm} \\ \text{Strong fermion parity} \quad & (-1)^{F^{\pm}}\psi_n^{\pm}(-1)^{F^{\pm}} = -\psi_n^{\pm} \quad (-1)^{F^{\pm}}\psi_n^{\mp}(-1)^{F^{\pm}} = \psi_n^{\mp}. \\ \text{Additional symmetry} \quad & \mathcal{R}\psi_n^{\pm}\mathcal{R}^{-1} = \psi_n^{\pm}, \quad \mathcal{R}z\mathcal{R}^{-1} = z^* \end{aligned} \tag{15}$$

- 多体のヒルベルト空間での代数:

$$\begin{aligned} & \mathcal{R}^2 = (-1)^{(N-1)/2} \\ & \mathcal{R}(-1)^{\mathcal{F}} = -(-1)^{\mathcal{F}}\mathcal{R} \\ & \mathcal{R}\mathcal{J} = (-1)^{(N-1)/2}\mathcal{J}\mathcal{R} \end{aligned}$$

TABLE I. Four-fold algebraic structure of total fermion parity symmetry $(-1)^{\mathcal{F}}$, modular conjugation symmetry $\mathcal{J}$, and antiunitary symmetry $\mathcal{R}$ in the double Hilbert space of open quantum fermionic systems. Here, $a \in \{+1, -1\}$ specifies the commutation or anticommutation relation between $(-1)^{\mathcal{F}}$ and $\mathcal{R}$ [i.e., $\mathcal{R}(-1)^{\mathcal{F}} = a(-1)^{\mathcal{F}}\mathcal{R}$]; $b \in \{+1, -1\}$ specifies the commutation or anticommutation relation between $\mathcal{J}$ and $\mathcal{R}$ [i.e., $\mathcal{R}\mathcal{J} = b\mathcal{J}\mathcal{R}$].

N(mod 4)	0	1	2	3
a	+1	-1	+1	-1
b	+1	+1	-1	-1
$\mathcal{R}^2$	+1	+1	-1	-1

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16 More extensive analysis

- q-body interactions and p-body dissipation

$$H_{\text{SYK}} = i^{q/2} \sum_{i_1 < \cdots < i_q} J_{i_1 \cdots i_q} \psi_{i_1} \cdots \psi_{i_q}$$

$$L_m = \sum_{i_1 < \cdots < i_p} K_{m, i_1, \cdots, i_p} \psi_{i_1} \cdots \psi_{i_p}$$

「パラメーター」: N, p, q, $K_{m,\boldsymbol{i}} K^*_{m,\boldsymbol{i}}$.

TABLE II. Periodic table of the Sachdev-Ye-Kitaev (SYK) Lindbladians with the linear dissipators $p = 1$ for $q \equiv 0, 2$ (mod 4) and the number N (mod 4) of Majorana fermions. See Appendix A for the detailed definitions of the symmetry classes. For the antiunitary symmetry $\mathcal{J}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$, the entries ± 1 mean the presence of the symmetry and its sign, and the entries 0 mean the absence of the symmetry. Additional symmetry can be present for $K_{m;i} K^*_{m;j} \in \mathbb{R}$. For odd $p \geq 3$, the antiunitary symmetry $\mathcal{P}, \mathcal{R}$ is no longer respected while the antiunitary symmetry $\mathcal{Q}, \mathcal{S}$ remains to be respected; for arbitrary odd $p \geq 3$ and N, we have class AI (or equivalently class D†) for $q \equiv 0$ with $K_{m;i} K^*_{m;j} \notin \mathbb{R}$ and $q \equiv 2$, and class BDI† for $q \equiv 0$ with $K_{m;i} K^*_{m;j} \in \mathbb{R}$.

N (mod 4)	0	1	2	3
fermion parity $(-1)^{\mathcal{F}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
modular conjugation $\mathcal{J}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{P}$	$+1$	0	-1	0
$\mathcal{Q}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{R}$	$+1$	0	-1	0
$\mathcal{S}$	$+1$	$+1$	$+1$	$+1$
$q \equiv 0$ (mod 4) [$K_{m;i} K^*_{m;j} \notin \mathbb{R}$]	AI = D†	AI = D†	AI = D†	AI = D†
$q \equiv 0$ (mod 4) [$K_{m;i} K^*_{m;j} \in \mathbb{R}$]	BDI† + $\mathcal{S}_{++}$	BDI†	CI† + $\mathcal{S}_{+-}$	BDI†
$q \equiv 2$ (mod 4)	BDI	AI = D†	CI	AI = D†

TABLE III. Periodic table of the Sachdev-Ye-Kitaev (SYK) Lindbladians with the even number p of dissipators for $q \equiv 0, 2$ (mod 4) and the number N (mod 4) of Majorana fermions. See Appendix A for the detailed definitions of the symmetry classes. For the antiunitary symmetry $\mathcal{J}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$, the entries ± 1 mean the presence of the symmetry and its sign, and the entries 0 mean the absence of the symmetry. Additional symmetry can be present for $K_{m;i} K^*_{m;j} \in \mathbb{R}$. For even N, we assume even fermion parity $(-1)^{\mathcal{F}} = +1$ in the double Hilbert space, which is relevant to the presence of modular conjugation symmetry $\mathcal{J}$ in the subspace with fixed fermion parity $(-1)^{F^{\pm}}$.

N (mod 4)	0	1	2	3
fermion parity $(-1)^{\mathcal{F}}$, $(-1)^{F^{\pm}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$
modular conjugation $\mathcal{J}$	$+1$	$+1$	0	$+1$
$\mathcal{P}$	$+1$	0	0	0
$\mathcal{Q}$	$+1$	$+1$	0	$+1$
$\mathcal{R}$	$+1$	0	0	0
$\mathcal{S}$	$+1$	$+1$	0	$+1$
$q \equiv 0$ (mod 4) [$K_{m;i} K^*_{m;j} \notin \mathbb{R}$], $q \equiv 2$ (mod 4)	AI=D†	AI=D†	A	AI=D†
$q \equiv 0$ (mod 4) [$K_{m;i} K^*_{m;j} \in \mathbb{R}$]	BDI†	BDI†	A + η = AIII	BDI†

Some observations:

- 周期は８でなく４になる。
- 「フリーな場合」では登場しなかった対称性クラスが現れる。(Only AZ † for quadratic Lindbladians)

- 全ての非エルミート対称性クラスが登場するわけではない
（相互作用効果とリンドブラディアンの制約の効果？）

- クラマース対称性は現れない（TR † with -1） (TR † with +1 はOK；shifted imaginary axis上でのクラマース縮退)
See also [Sa-Ribiro-Prosen (22)]
- 準位統計
- トポロジーや量子異常との関係？

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17 有限時間ダイナミクスとDissipative form factor

- Dissipative form factor [Can 19]

$$F(T_L) := \overline{\mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}^*} e^{T_L\mathcal{L}}} \equiv e^{iN\mathcal{S}(T_L)}$$

- 環境との相互作用を切る極限でスペクトルフォームファクターに一致:
$$\mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}^*} e^{T_L\mathcal{L}} = |\mathrm{Tr}_{\mathcal{H}} e^{-iT_L H}|^2$$
- Loschmidt echo (return probability): $\mathrm{Tr}_{\mathcal{H}}\left[\rho(T_L)\rho(0)\right]$
  - 初期状態の平均を取る： $\rho(0) \to \rho_U = U\rho_0 U^\dagger$ and Haar average over U
$$\int dU \langle \rho_U | e^{t\mathcal{L}} | \rho_U \rangle = \frac{\mathrm{Tr}(\rho_0^2) - 1/L}{L^2-1} \mathrm{Tr}\, e^{t\mathcal{L}} + \frac{L - \mathrm{Tr}(\rho_0^2)}{L^2-1}$$
- デコヒーレンスレート：
$$D = -\frac{2\,\mathrm{Tr}[\rho(0)(d\rho(t)/dt)]}{\mathrm{Tr}\left[\rho(0)^2\right]}\Bigg|_{t=0},$$
$$D_{\mathrm{av}} = -\frac{2}{\mathrm{Tr}\left[\rho_0^2\right]} \frac{d}{dt} \int dU\, \mathrm{Tr}[\rho_U(0)\rho_U(t)]\Bigg|_{t=0} \tag{16}$$
- DFFはラージN極限を使って計算できる

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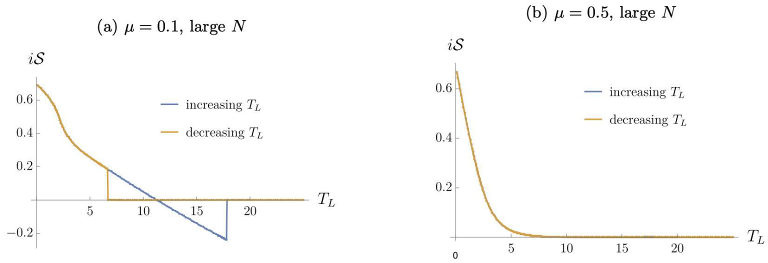
18 DFF in large N

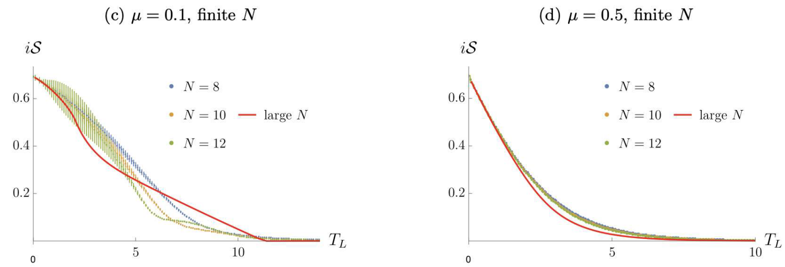
$$\mathcal{L} = -iH^+_{\mathrm{SYK}} + i(-1)^{\frac{q}{2}} H^-_{\mathrm{SYK}} - i\mu \sum_i \psi^i_+ \psi^i_- - \mu \frac{N}{2} \rrbracket$$

- Large N analysis:
$$\begin{aligned} F(T_L) &= \mathrm{Tr}\left[e^{T_L\mathcal{L}}\right] \\ &= \int \mathcal{D}[\psi^+, \psi^-] e^{iS[\psi^+,\psi^-]} \\ &= \int \mathcal{D}[\Sigma, G] e^{iS[\Sigma,G]} \end{aligned} \tag{17}$$

  Collective field:
$$G_{\alpha\beta}(t_1,t_2) = \frac{-i}{N} \sum_n \psi^\alpha_n(t_1) \psi^\beta_n(t_2) \tag{18}$$
- For small μ, a first order dynamical phase transition; similar to Hawking-Page
- 2nd order transition at small T_L
- For large μ, no transition

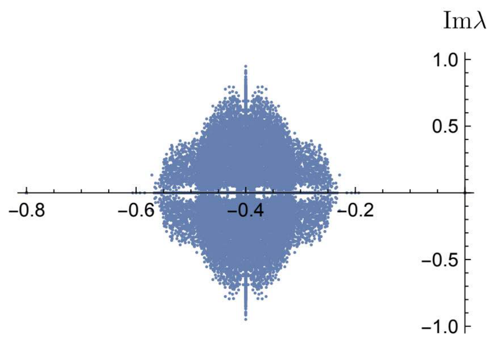


- 有限Nとの比較

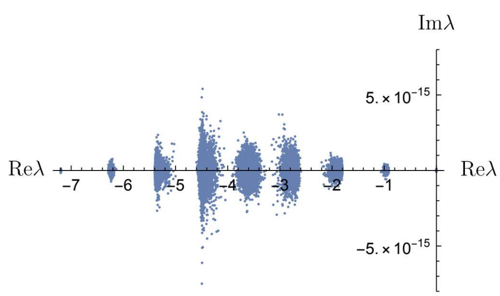


- スペクトルの変化

(a) $\mu = 0.1$



(d) $\mu = 0.9$



19 Comparison with two-coupled SYK model

TABLE I: Analogy between the two-coupled SYK model [58] and the SYK Lindbladian with the non-random linear jump operators for the case of small coupling μ. The wormhole in the coupled SYK model corresponds to the thermofield double (TFD) state at a certain temperature determined by μ. Similarly, for the Lindbladian SYK model, the late time solution corresponds to the infinite temperature TFD state, which is the stationary state of the Lindbladian.

Two-coupled SYK model	SYK Lindbladian with the non-random linear dissipators
Left/Right system	Bra (+)/Ket (−) contour
$H = H^L_{\text{SYK}} + (-1)^{\frac{q}{2}} H^R_{\text{SYK}} + i\mu \sum_i \psi^i_L \psi^i_R$	$\mathcal{L} = -iH^+_{\text{SYK}} + i(-1)^{\frac{q}{2}} H^-_{\text{SYK}} - i\mu \sum_i \psi^i_+ \psi^i_- - \mu \frac{N}{2}\mathbb{I}$
Inverse temperature β	Periodicity T_L
Partition function $\text{Tr}\left(e^{-\beta H}\right)$	Dissipative form factor $\text{Tr}\left(e^{T_L \mathcal{L}}\right)$
Energy $E = \langle H \rangle$	Lindbladian $\langle \mathcal{L} \rangle$ (average decoherence rate)
Specific Heat $C = \langle H^2 \rangle - \langle H \rangle^2$	
Energy gap	Decay rate
Black hole	Early time complex solution
Wormhole (TFD)	Late time real solution (Infinite temperature TFD)
Hawking-Page transition	Late time first-order transition (real-complex spectral transition?)
N/A	Early time second-order order transition
Real time physics (e.g. chaos exponents)	?

20 Summary

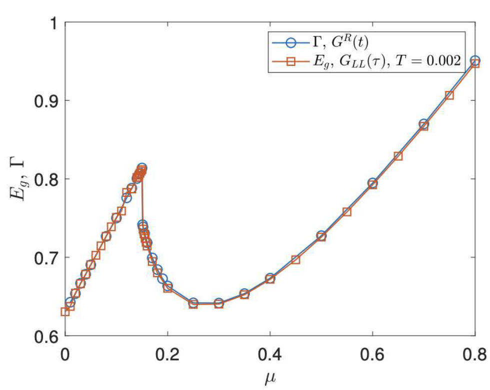
- 多体開放量子系の解ける模型として、SYK Lindbladianを導入した。
- SYK Lindbladian の対称性を分類し、（フェルミオニックな）Lindbladianに現れうる対称性を分類した
- トポロジーや量子異常との関係？
- Physical relevance?
  - Graphene flake, Cold atoms, quantum computers, cavity QED [Uhrich 23]
- Maybe also interesting for quantum gravity community

21 Stationary properties

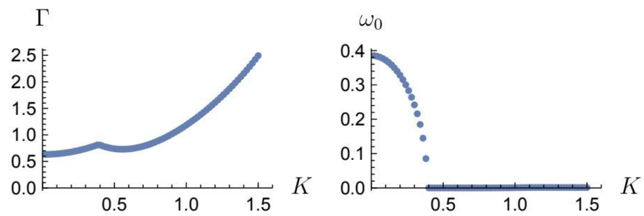
Large N solution to the stationary Green function:

$$G^R(t) \approx A e^{-\Gamma t} \sin(\omega_0 t + \phi)$$

Large N, (q,p)=(4,1) [Kulkarni-Numasawa-SR (21), Sa-Ribeiro-Prose (21), García-García-Sá-Verbaarschot-Zheng (22)]



Large N, (q,p)=(4,2) K: randomess in the jump operator



- Stationary solutions, $G_{\pm\pm}(t_1, t_2) = G_{\pm\pm}(t_1 - t_2)$.
- Highly non-monotonic behavior of the decay rate $\mathbf{\Gamma}$
  (consequence of the competition between dissipations and interactions?)
- Phase transitions between fully damped and coherent dynamics as we change $\boldsymbol{\mu}$ or $\boldsymbol{K}$
- Large $\boldsymbol{q}$ analysis is also possible

対称性クラスの等価性

TABLE XIII. Equivalence between the real AZ symmetry class with sublattice symmetry (SLS) and the real AZ† symmetry class with SLS. The subscript of $\mathcal{S}_\pm$ specifies the commutation (+) or anticommutation (−) relation to TRS/TRS† and/or PHS/PHS†. For the symmetry classes that involve both TRS/TRS† and PHS/PHS† (BDI, DIII, CII, and CI; BDI†, DIII†, CII†, and CI†), the first subscript specifies the relation to TRS/TRS†, and the second one specifies the relation to PHS/PHS†.

AZ† class	$\mathcal{S}_+$	$\mathcal{S}_-$	$\mathcal{S}_{++}$	$\mathcal{S}_{+-}$	$\mathcal{S}_{-+}$	$\mathcal{S}_{--}$
AI†	D + $\mathcal{S}_+$	C + $\mathcal{S}_-$				
BDI†			BDI + $\mathcal{S}_{++}$	DIII + $\mathcal{S}_{-+}$	CI + $\mathcal{S}_{+-}$	CII + $\mathcal{S}_{--}$
D†	AI + $\mathcal{S}_+$	AII + $\mathcal{S}_-$				
DIII†			CI + $\mathcal{S}_{++}$	CII + $\mathcal{S}_{-+}$	BDI + $\mathcal{S}_{+-}$	DIII + $\mathcal{S}_{--}$
AII†	C + $\mathcal{S}_+$	D + $\mathcal{S}_-$				
CII†			CII + $\mathcal{S}_{++}$	CI + $\mathcal{S}_{-+}$	DIII + $\mathcal{S}_{+-}$	BDI + $\mathcal{S}_{--}$
C†	AII + $\mathcal{S}_+$	AI + $\mathcal{S}_-$				
CI†			DIII + $\mathcal{S}_{++}$	BDI + $\mathcal{S}_{-+}$	CII + $\mathcal{S}_{+-}$	CI + $\mathcal{S}_{--}$

TABLE XIV. Equivalence between pseudo-Hermiticity and sublattice symmetry as an additional symmetry in the AZ symmetry class. For the complex classes, the subscript of $\eta_\pm$ and $\mathcal{S}_\pm$ specifies the commutation (+) or anticommutation (−) relation to chiral symmetry. For the real classes, the subscript of $\eta_\pm$ and $\mathcal{S}_\pm$ specifies the commutation (+) or anticommutation (−) relation to time-reversal symmetry (TRS) and/or particle-hole symmetry (PHS). For the symmetry classes that involve both TRS and PHS (BDI, DIII, CII, and CI), the first subscript specifies the relation to TRS and the second one to PHS.

AZ class	η	η_+	η_-	η_{++}	η_{+-}	η_{-+}	η_{--}
A	AIII						
AIII		AIII + $\mathcal{S}_+$	AIII + $\mathcal{S}_-$				
AI		BDI†	DIII†				
BDI				BDI + $\mathcal{S}_{++}$	BDI + $\mathcal{S}_{-+}$	BDI + $\mathcal{S}_{+-}$	BDI + $\mathcal{S}_{--}$
D		BDI	DIII				
DIII				DIII + $\mathcal{S}_{--}$	DIII + $\mathcal{S}_{+-}$	DIII + $\mathcal{S}_{-+}$	DIII + $\mathcal{S}_{++}$
AII		CII†	CI†				
CII				CII + $\mathcal{S}_{++}$	CII + $\mathcal{S}_{-+}$	CII + $\mathcal{S}_{+-}$	CII + $\mathcal{S}_{--}$
C		CII	CI				
CI				CI + $\mathcal{S}_{--}$	CI + $\mathcal{S}_{+-}$	CI + $\mathcal{S}_{-+}$	CI + $\mathcal{S}_{++}$