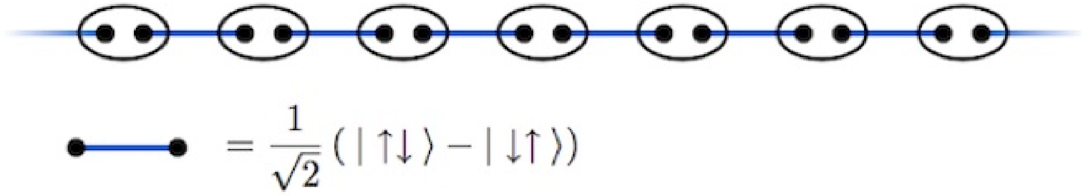

Figure 8: *Graphical representation of the AKLT ground state. The black dots denote auxiliary spin 1/2 "particles", the ovals project on spin 1, and the lines mean that two spins 1/2 form a singlet state. Since two of the four spin 1/2 on two adjacent sites form a singlet, the maximal total spin is 1, so the projection on spin 2 gives zero. Since this is true for all pairs, this state is clearly an eigenstate of H_{AKLT}. Also note the unpaired spins at the end of the chain, which are the fractionalized edge modes. (Figure taken from the Wikipedia article: AKLT model.)*

We now complement the above rather abstract argument, which also relied on the assumption of large S, with a description of a very instructive and exactly solvable model for $S = 1$, which is a close cousin of the Heisenberg model in Eq. (1). The Hamiltonian for this AKLT chain, named after its inventors Ian Affleck, Tom Kennedy, Elliott Lieb and Hal Tasaki, is [1]

$$H_{AKLT} = \sum_i \left[\frac{1}{2}\vec{S}_i \cdot \vec{S}_{i+1} + cfrac16\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 + \frac{1}{3}\right] = \sum_i P_2(\vec{S}_i + \vec{S}_{i+1}) \tag{18}$$

where $\vec{S}_i$ is a spin 1 operator at the lattice site i, and P_2 projects on the subspace corresponding to spin 2 on two adjacent lattice sites. To find the ground state, we imagine that each link in the chain hosts two auxiliary spin 1/2 that are projected to a spin 1. As explained in Fig. 8, forming a spin singlet at each link in the chain gives an eigenstate of the Hamiltonian with zero energy. Since the Hamiltonian is a sum of projectors, the ground state energy has to be non-negative and we conclude that we have constructed a ground state of the full interacting model. From the figure we also see that there are two "unpaired" spin 1/2 degrees of freedom at the two ends of the chain, which is an example of *quantum number fractionalization*, since the original degrees of freedom were spin 1! One can show that the unpaired spins give rise to a double degeneracy of the ground state, but the most striking property of the AKLT chain is that it has a *Haldane gap*, as was shown analytically in a later article by the same authors [2].

Later work has greatly deepened our understanding of the Haldane phase of the Heisenberg antiferromagnetic chain. Although there is no local order parameter, it is sometimes possible to characterise it by a non-local *string order parameter* [30] introduced earlier in the context of statistical mechanics [16].

[1] Ian Affleck, Tom Kennedy, Elliott H Lieb, and Hal Tasaki. Rigorous results on valence-bond ground states in antiferromagnets. *Physical Review Letters*, 59(7):799, 1987.

[2] Ian Affleck, Tom Kennedy, Elliott H Lieb, and Hal Tasaki. Valence bond ground states in isotropic quantum antiferromagnets. In *Condensed Matter Physics and Exactly Soluble Models*, pages 253–304. Springer, 1988.

[30] Tom Kennedy and Hal Tasaki. Hidden $z_2 \times z_2$ symmetry breaking in Haldane gap antiferromagnets. *Physical Review B*, 45(1):304, 1992.