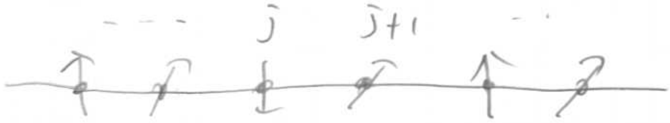
<motivation>

III) Haldane "conjecture"



antiferromagnetic quantum Heisenberg chain

$$H = \sum_{j=1}^{L} \mathbf{S}_j \cdot \mathbf{S}_{j+1}$$

$$\mathbf{S}_j = (S_j^{(x)}, S_j^{(y)}, S_j^{(z)}), \quad \mathbf{S}_j^2 = S(S+1)$$

Haldane 1983

$S = \frac{1}{2}, \frac{3}{2}, \cdots$

i) the g.s. is unique

ii) no gap above the g.s. energy (as $L \uparrow \infty$)

iii) the g.s. correlation shows a power-law decay

$S = 1, 2, 3, \cdots$

i') the g.s. is unique

ii') $\exists$ a gap above the g.s. energy (Haldane gap)

iii') the g.s. correlation shows exponential decay

unique disordered g.s. with a gap

SURPRISING!! (in 1980's)

From now on

$\boxed{\text{We } \cancel{\text{only}} \text{ (mainly) consider } S=1 \text{ chains}}$

$S_j^{(\alpha)}$ 3×3 matrix

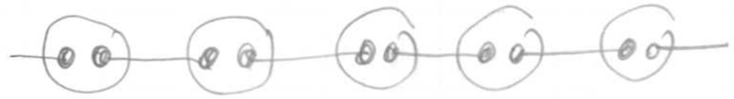
▷ an $S=1$ chain with a unique gapped g.s.

<u>AKLT model 1987</u>

$$H_{\mathrm{AKLT}} = \sum_{j=1}^{L} \left\{ \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \frac{1}{3} (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 \right\}$$

i'), ii'), iii') are proved rigorously → a prototypical model in the "Haldane phase"

<u>exact g.s.</u> (valence-bond solid (VBS) state)

$|\mathrm{VBS}\rangle =$ 

with $\overset{1}{\bullet}\!\!-\!\!\overset{2}{\bullet} = \frac{1}{\sqrt{2}} \{ |\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \}$

spin singlet of two $S=\frac{1}{2}$'s

 ← symmetrization

$$\underbrace{|\sigma\rangle|\sigma'\rangle}_{\text{two } S=\frac{1}{2}\text{'s}} \longrightarrow \underbrace{\frac{1}{2}(|\sigma\rangle|\sigma'\rangle + |\sigma'\rangle|\sigma\rangle)}_{\text{a state with } S=1}$$

▷ a trivial $S=1$ model with a unique gapped g.s.

$$H_{\mathrm{trivial}} = \sum_{j=1}^{L} (S_j^{(z)})^2.$$

$$\left(\begin{array}{l} S_j^z |0\rangle_j = 0 \\ S_j^z |\pm\rangle_j = \pm |\pm\rangle_j \end{array} \right)$$

the g.s. $= \bigotimes_{j=1}^{L} |0\rangle_j$

gap $= 1$. ← trivial

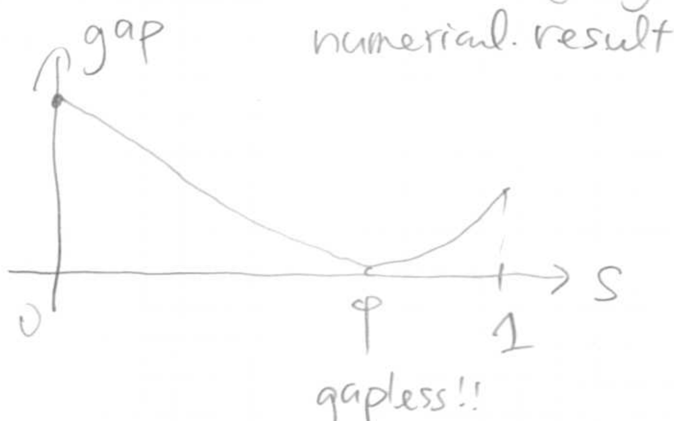
Is this also Haldane gap??

topological phase transition

$s \in [0,1]$

$$H_s = s H_{AKLT} + (1-s) H_{trivial}$$

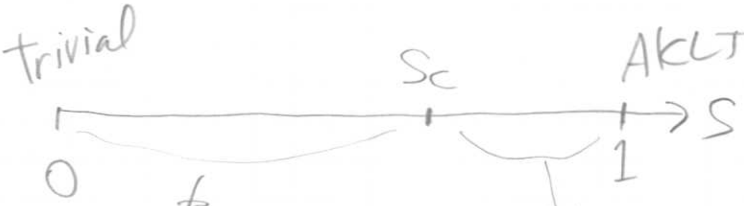
does this have a unique gapped g.s. for all s?



Th. (Tasaki 2018.)

$\exists s \in (0,1)$ at which the model either

- is gapless
- has more than two g.s.
- exhibits a discontinuity in the g.s. expec. value



unique gapped g.s.
no symmetry breaking
exp. decaying correlations

unique gapped g.s.
no symmetry breaking
exp. decaying correlations

cannot be distinguished by a (local) orderparameter

"topological" phase transition

MAIN QUESTIONS

- Is there a phase to which AKLT belongs?
- If so, what is the universal characterization of the phase?

cf. the phases of classical matter

solid phase : spontaneous breakdown of translational symmetry
fluid phase : no symmetry breaking

Translation invariance of the system is necessary for the robustness of the solid phase.

$\boxed{\text{the solid phase is "protected" by the translation symmetry}}$

a recent review of SSB
Beekman et al.

<Some math about symmetry>

III (projective) representations of a group

G : a finite group

- unitary matrices U_g with $g \in G$ form a (genuine) representation of G iff

  identity

  - $U_e = 1$
  - $U_g U_h = U_{gh}$ for $\forall g, h \in G$

- unitary matrices U_g with $g \in G$ form a projective representation of G iff
  - $U_e = 1$
  - $U_g U_h = \omega(g,h) U_{gh}$ for $\forall g, h \in G$

    with some phase factor $\omega(g,h) \in \mathbb{C}$, $|\omega(g,h)| = 1$

- Two projective reps. $(U_g)_{g\in G}$ and $(U'_g)_{g\in G}$ are equivalent if $U_g = \psi(g)\, U'_g$ for $\forall g \in G$

  with some phase factor $\psi(g) \in \mathbb{C}$, $|\psi(g)| = 1$

the equivalence classes of proj. reps.

$$\cong H^2(G, U(1))$$

(the 2nd group cohomology)

(iii) The group $\mathbb{Z}_2 \times \mathbb{Z}_2 = D_2$ → dihedral group

abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, x, y, z\}$ BAD NOTATION inspired by quantum information notation of Pauli matrices

$$x^2 = y^2 = z^2 = e$$
$$xy = yx = z, \quad \text{etc.}$$

$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$

<u>Th.</u> there are two equivalence classes of proj. reps. of $\mathbb{Z}_2 \times \mathbb{Z}_2$

- trivial: equivalent to a genuine rep. $\Leftrightarrow U_\alpha U_\beta = U_\beta U_\alpha$
- nontrivial: not eq. to a gen. rep. $\Leftrightarrow U_\alpha U_\beta = -U_\beta U_\alpha$

$\alpha, \beta \in \{x, y, z\}$
$\alpha \neq \beta$

proj. rep. in terms of QM angular momentum

$$\mathbb{J} = (J^{(x)}, J^{(y)}, J^{(z)}), \quad \mathbb{J}^2 = J(J+1).$$

$$U_e = 1, \quad U_\alpha = \exp[-i\pi J^{(\alpha)}]$$

- trivial if J is an integer
- nontrivial if J is a half-odd-integer.

$\because$
$$\left(U_x U_z U_x^* = U_x e^{-i\pi J^{(z)}} U_x^* = e^{-i\pi \overset{-J^{(z)}}{\overbrace{U_x J^{(z)} U_x^*}}} = U_z^* \right.$$

$$\left. \therefore U_x U_z = U_z^* U_x = (U_z^*)^2 U_z U_x = \begin{cases} U_z U_x \\ -U_z U_x \end{cases} \right)$$

(in particular
$J = \frac{1}{2} \quad U_\alpha = -i\sigma_\alpha \quad \alpha \in \{x, y, z\}$)

A^* denotes the adjoint (Hermitian conjugate) of an operator or a matrix

⟨Symmetry Protected Topological (SPT) phases⟩

Can we connect H_{AKLT} and $H_{trivial}$ continuously via models with a unique gapped g.s. ?

more precisely

is there H_s which continuously depends on $s \in [0,1]$ s.t. (→ short ranged Ham.)

$$\begin{cases} H_s \text{ has a unique g.s. for all } s \in [0,1] \\ H_0 = H_{trivial}, \quad H_1 = H_{AKLT} \end{cases}$$?

Yes if any short ranged Hamitonians are allowed!

Chen, Gu, Wen 2011, Ogata 2016, 2017 (↑ RIGOROUS!)

No if some symmetry is imposed on H_s.

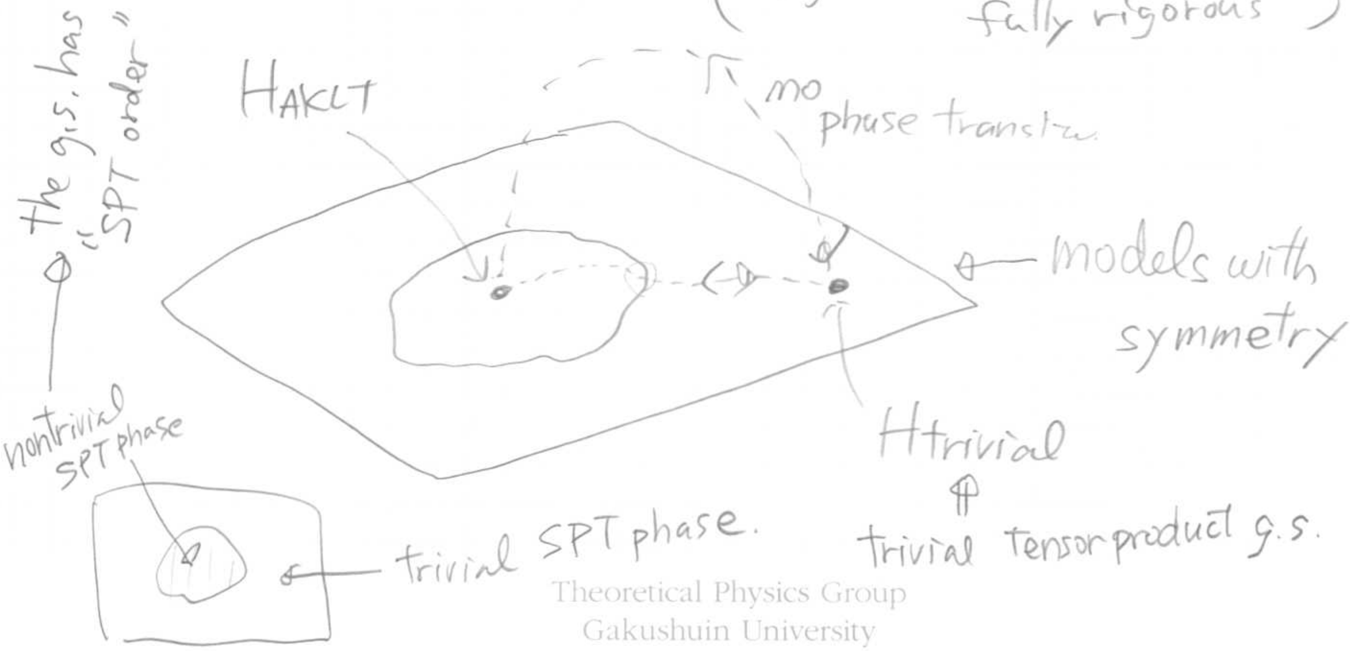
H_{AKLT} is in a nontrivial SPT phase Gu, Wen 2009.

one of

- $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry
- time-reversal symmetry
- bond-centered inversion symmetry

← Pollmann, Turner, Berg, Oshikawa 2010 2012. (↓ PTBO)

(Ogata 2018, 2019 fully rigorous)



⟨$\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant models⟩

$\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation for a spin chain

$$U_e = 1, \quad U_\alpha = \exp\Big[-i\pi \sum_{j=1}^{L} S_j^{(\alpha)}\Big] \qquad \alpha \in \{x, y, z\}$$

↳ π-rotation about α-axis

for $\alpha, \beta \in \{x, y, z\}$

$$U_\alpha^* S_j^{(\beta)} U_\alpha = \begin{cases} S_j^{(\beta)} & \beta = \alpha \\ -S_j^{(\beta)} & \beta \neq \alpha \end{cases}$$

Assumptions

- Hamiltonian H $\begin{cases} \text{short ranged} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ invariant } U_\alpha^* H U_\alpha = H \quad \alpha \in \{x, y, z\} \end{cases}$

- H has a unique g.s. with a gap

then $U_\alpha |GS\rangle = C_\alpha |GS\rangle$ $\qquad \alpha \in \{x, y, z\}$

$C_\alpha = \pm 1$

examples $H_{AKLT} = \sum \left(\mathbf{S}_j \cdot \mathbf{S}_{j+1} + \frac{1}{3} (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 \right)$

$H_{trivial} = \sum (S_j^z)^2$

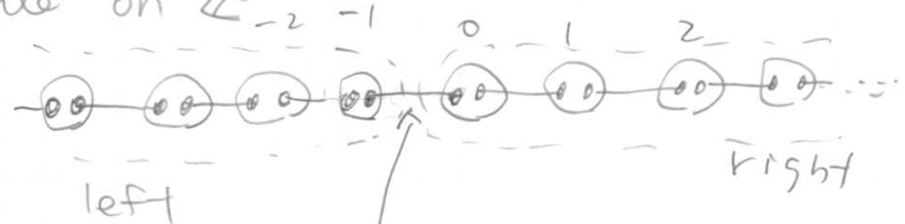
⟨entanglement and $\mathbb{Z}_2$ index for SPT phases⟩

formal (= non-rigorous) consideration for spin chains on the infinite chain $\mathbb{Z}$.

decomposition $\mathbb{Z} = \{\cdots, -2, -1\} \cup \{0, 1, 2, \cdots\}$

left half-infinite chain, right half-infinite chain

▷ VBS state on $\mathbb{Z}$

$|VBS\rangle = \cdots$ 

left and right halves are entangled by the singlet  $= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$

$= \frac{1}{\sqrt{2}}\Big\{$ 

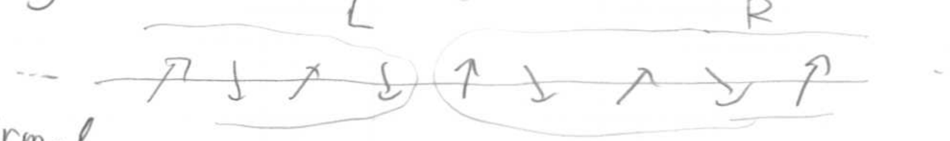
$-$  $\Big\}$

$$= \frac{1}{\sqrt{2}}\left(|\Phi_\uparrow\rangle_L |\Psi_\downarrow\rangle_R - |\Phi_\downarrow\rangle_L |\Psi_\uparrow\rangle_R\right)$$

(formal) Schmidt decomposition

$|\Psi_\uparrow\rangle_R, |\Psi_\downarrow\rangle_R$ may be (effectively) regarded as states with $S = \frac{1}{2}$.

III) general $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant unique gapped g.s. on $\mathbb{Z}$.



formal

• Schmidt decomposition

$$|GS\rangle = \sum_{j}^{J_R} \sqrt{p_j}\, |\Phi_j\rangle_L |\Psi_j\rangle_R \quad , \; p_j > 0, \; \sum_j p_j = 1$$

J_R — small number

reduced density matrix on the right half

$$\rho_R = \mathrm{Tr}_L \left[|GS\rangle\langle GS| \right] = \sum_j p_j \, |\Psi_j\rangle_R \, {}_R\langle\Psi_j|$$

half-odd-integer effective spins

• assume that (as in the VBS)

$|\Psi_j\rangle_R$ are (effectively) states with half-odd-integer spins.

U_g : the action of $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ (π-rotations) on the right half

then $U_\alpha U_\beta = - U_\beta U_\alpha$ for $\alpha, \beta \in \{x, y, z\}$, $\alpha \neq \beta$

• $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance of $|GS\rangle$

$\Downarrow$

$$U_g \rho_R U_g^* = \rho_R \quad \text{i.e. } [U_g, \rho_R] = 0 \text{ for } \forall g \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

1) we can assume $U_z |\Psi_j\rangle_R = c_j |\Psi_j\rangle_R$.

2) let $|\Psi'_j\rangle_R = U_x |\Psi_j\rangle_R$.

then
$$U_z |\Psi'_j\rangle_R = U_z U_x |\Psi_j\rangle_R = - U_x U_z |\Psi_j\rangle_R$$
$$= - c_j |\Psi'_j\rangle_R$$

$\therefore$

${}_R\langle\Psi_j|\Psi'_j\rangle_R = 0$

now: $\rho_R = U_x \rho_R U_x^*$

$\Downarrow$

$$\sum_j p_j |\Psi_j\rangle_R {}_R\langle\Psi_j| = \sum_j p_j |\Psi'_j\rangle_R {}_R\langle\Psi'_j|$$

since ${}_R\langle\Psi_j|\Psi'_j\rangle_R = 0$ all p_j must come in pairs!

any e.v. p_j is even-fold degenerate

$\downarrow$ at least two-fold deg.

example.

$$\left(\rho_R^{VBS} = \sum_{\sigma=\uparrow,\downarrow} \frac{1}{2} |\Psi_\sigma\rangle_R {}_R\langle\Psi_\sigma| \right)$$

$S_{LR} = \log 2$

$$\therefore S_{LR} = -\sum_j p_j \log p_j \geq \log 2 \quad \leftarrow \text{PTBO}$$

symmetry $\rightarrow$ lower bound on S_{LR}.

"entanglement imposed by symmetry"

$\uparrow$ an universal characterization of SPT order

integer/effective spins

if $|\Psi_j\rangle_R$ are states with integer spins.

There is no such lower bound for S_{LR}

(we can "turn off" S_{LR})

$$\left(\rho_R^{\text{trivial}} = |\text{all } 0\rangle_R {}_R\langle \text{all } 0| \qquad S_{LR} = 0 \right)$$

III $\mathbb{Z}_2$-index $\quad \sigma = \pm 1$

U_g : action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the half-infinite chain

for $\alpha, \beta \in \{x, y, z\}$, $\alpha \neq \beta$

$$U_\alpha U_\beta = \underset{\sim}{\sigma}\, U_\beta U_\alpha$$

$\sigma = -1 \Rightarrow S_{LR} \geq \log 2$ nontrivial SPT order

III BUT

all these ~~were~~ one very formal consideration for states on the infinite chain $\mathbb{Z}$.

- large finite chain

system of $\frac{L}{2}$ spins with $S=1$

total spin is always integer

no way to have $\sigma = -1$ ...

- How can we define σ ?

  - MPS $\quad$ PTBO 2010, 2012

  - operator algebra $\quad$ Ogata 2018

    $\hookrightarrow$ DAY 2

⟨Matrix Product States (MPS)⟩ → FNW

Fannes, Nachtergaele, Werner 1989, 1992 CMP main.

Ⅲ translation invariant MPS

spin S chain on $\{1,2,\dots,L\}$

- standard basis states $|s_1,\dots,s_L\rangle = \bigotimes_{j=1}^{L} |s_j\rangle_j$

$$S_j^{(z)}|s\rangle_j = s|s\rangle_j \ , \ s=-S,\dots,S$$

- $D\times D$ (fix) matrices M^s with $s=-S,\dots,S$

- MPS

$$|\Phi\rangle = \sum_{s_1,\dots,s_L=-S}^{S} \underbrace{\mathrm{Tr}[M^{s_1}\cdots M^{s_L}]}_{\text{coefficients}} |s_1,\dots,s_L\rangle$$

  - a compact way of writing down a quantum state
  - ◎ states with small entanglement (area law states) can be well approximated by MPS.

Ⅲ Examples

$S=1$ VBS state MPS with $D=2$

$$M^+ = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix},\ M^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ M^- = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$$

$S=1$ trivial state $|0,00\cdots0\rangle$ MPS with $D=1$

$$M^+ = M^- = (0),\ M^0 = (1)$$

trivial.

III injective MPS

- an important and useful class of MPS → uniqueness theorem FNW 92
- corresponds to a state with small entanglement which is not a Schrödinger's cat.
- The two examples are both injective

Def. $|\Phi\rangle$ is injective iff (or premitive.)

(i) $\sum_{s=-S}^{S} M^s (M^s)^* = \lambda I$ with $\lambda > 0$

(ii) $\exists \ell$ s.t. $M^{s_1} \cdots M^{s_\ell}$ with all possible $s_1, \cdots, s_\ell$ span the wholespace of $D \times D$ matrices

(Rem. (ii) $\Rightarrow$ the map $W \mapsto \sum_{s_1, \cdots, s_\ell} \mathrm{Tr}[W M^{s_1} \cdots M^{s_\ell}] |s_1, \cdots, s_\ell\rangle$ is injective)

<index theory of PTBO 2010, 2012>

very close idea → Pere-Garcia, Wolf, Sanz, Verstraete, Cirac 2008

Matsui 2001 ← proj. rep. in MPS and more!

III Consequence of on-site symmetry

unitary $U = \bigotimes_{j=1}^{L} u_j$, u_j copy of a unitary u acting on a single spin.

$S=1$ injective MPS

$$|\Phi\rangle = \sum_{\$}{}^{'} \mathrm{Tr}[M^{s_1} \cdots M^{s_L}]\,|\$\rangle$$

$$\$ = (s_1, \cdots, s_L) \ , \quad s_j = 0, \pm 1$$

$$U|\Phi\rangle = \sum_{\$}{}^{'} \mathrm{Tr}[M^{s_1} \cdots M^{s_L}]\, U|\$\rangle$$

$\sum_{\$'} |\$'\rangle\langle\$'|$ $\quad \langle\$'|U|\$\rangle = \prod_{j=1}^{L} \langle s_j'|u|s_j\rangle$

$$= \sum_{\$,\$'}{}^{'} \mathrm{Tr}[M^{s_1} \cdots M^{s_L}]\,|\$'\rangle \prod_{j=1}^{L} \langle s_j'|u|s_j\rangle$$

$$= \sum_{\$}{}^{'} \mathrm{Tr}[\tilde{M}^{s_1} \cdots \tilde{M}^{s_L}]\,|\$\rangle$$

with $$\tilde{M}^{s} = \sum_{s'=0,\pm 1}{}^{'} \langle s|u|s'\rangle M^{s'} \quad \cdots\ (*)$$

- assume the invariance of $|\Phi\rangle$

$$U|\Phi\rangle = e^{i\eta}|\Phi\rangle \text{ with some } \eta \quad \cdots (\star)$$

obvious $\Uparrow$ $\Downarrow$ ← uniqueness theorem of FNW (for injective MPS)

$\tilde{U}$ is unique up to a phase.

$$\tilde{M}^s = e^{i\zeta}\,\tilde{U}^* M^s \tilde{U} \quad \cdots (**) \quad \text{for } s=0,\pm 1$$

with a constant ζ and unitary $D\times D$ matrix $\tilde{U}$

(ζ, $\tilde{U}$ indep. of s, L)

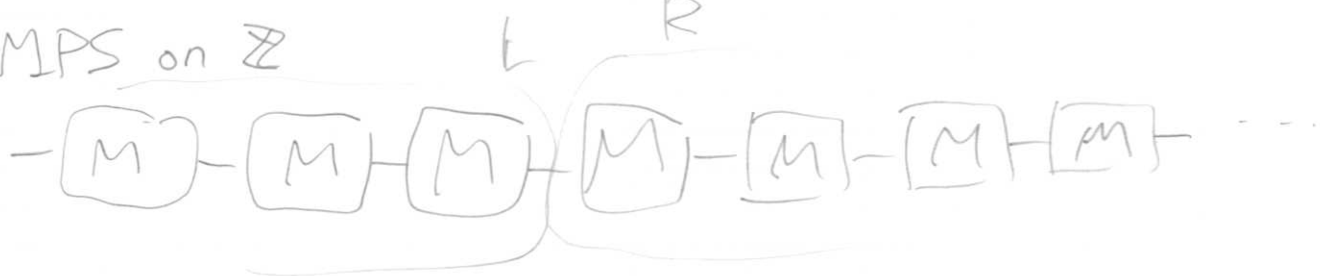
from $(\star)$, $(**)$

$$\boxed{M^s = e^{i\zeta}\sum_{s'=0,\pm1}\langle s|u^*|s'\rangle\,\tilde{U}^* M^{s'}\tilde{U}}$$

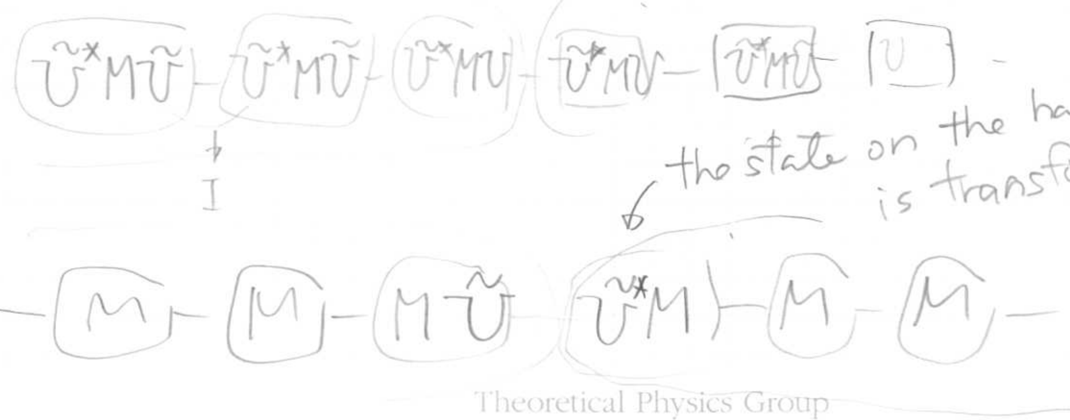
strong constraint on M^s that comes from $(\star)$

<u>physical meaning of $(**)$</u>

MPS on $\mathbb{Z}$



apply U



the state on the half-inf. chain is transformed by $\tilde{U}^*$!!

you don't see $\tilde{U}$ at ∞

(III) proj. rep. of $\underset{\uparrow\text{ always}}{G=\mathbb{Z}_2\times\mathbb{Z}_2}$ and $\mathbb{Z}_2$-index of PTBO's

$|\Phi\rangle$ $S=1$ injective MPS $\leftarrow$ (the unique gapped g.s. of a $\mathbb{Z}_2\times\mathbb{Z}_2$ invariant spin chain)

- assume $\mathbb{Z}_2\times\mathbb{Z}_2$ invariance

$$U_g|\Phi\rangle = e^{i\eta_g}|\Phi\rangle \quad \text{for } g\in G$$

then $\exists\, \varsigma_g, \tilde{U}_g$ (indep. of s, L) $\rightarrow$ (of course $\varsigma_e=0$, $\tilde{U}_e=I$)

$$\boxed{M^s = e^{i\varsigma_g}\sum_{s'=0,\pm1}\langle s|U_g^*|s'\rangle\, \tilde{U}_g^* M^{s'}\tilde{U}_g} \quad \text{for } g\in G \quad \text{— (0)}$$

$$U_e = 1,\quad U_\alpha = \exp[-i\pi \underset{\uparrow\text{ single spin op.}}{S^{(\alpha)}}] \quad \alpha\in\{x,y,z\}$$

$\uparrow$ genuine rep. of $\mathbb{Z}_2\times\mathbb{Z}_2$ (~~S: integer~~) $(S=1)$

(0) with $g\to h$

$$M^s = e^{i\varsigma_h}\sum_{s''}\langle s|U_h^*|s''\rangle\, \tilde{U}_h^* \underset{\uparrow (0)}{M^{s''}}\tilde{U}_h$$

$$= e^{i\varsigma_h}\sum_{s''}\langle s|U_h^*|s''\rangle\, \tilde{U}_h^*\Big\{e^{i\varsigma_g}\sum_{s'}\langle s''|U_g^*|s'\rangle$$

$$\tilde{U}_g^* M^{s'}\tilde{U}_g\Big\}\tilde{U}_h$$

$$= e^{i(\varsigma_g+\varsigma_h)}\sum_{s'}\langle s|U_{gh}^*|s'\rangle\,(\tilde{U}_g\tilde{U}_h)^* M^{s'}\,\tilde{U}_g\tilde{U}_h$$

(0) with $g\to gh$

$$M^s = e^{i\varsigma_{gh}}\sum_{s'}\langle s|U_{gh}^*|s'\rangle\,\tilde{U}_{gh}^* M^{s'}\tilde{U}_{gh}$$

$$(U_{gh} = U_g U_h \Rightarrow \xi_{gh} = \xi_g + \xi_h)$$

$$(\tilde{U}_g \tilde{U}_h)^* M^s \tilde{U}_g \tilde{U}_h = \tilde{U}_{gh}^* M^s \tilde{U}_{gh}$$

$$\downarrow$$

$$[W, M^s] = 0 \quad \text{for } \forall s = 0, \pm 1 \text{ with } W = \tilde{U}_g \tilde{U}_h \tilde{U}_{gh}^*$$

$$\downarrow$$

$$[W, M^{s_1} M^{s_2} \cdots M^{s_\ell}] = 0 \quad \text{for } \forall s_1, \cdots, s_\ell$$

$\downarrow$ injectivity

$$[W, A] = 0 \quad \text{for } \forall D \times D \text{ matrix } A$$

$$\downarrow$$

$$W = \omega I, \quad \omega \in \mathbb{C}, \ |\omega| = 1$$

we find $\quad \tilde{U}_g \tilde{U}_h = \omega(g,h) \tilde{U}_{g,h} \quad$ for $\forall g, h \in G$

$\tilde{U}_g$ with $g \in G$ form a proj. rep. of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

moreover $\tilde{U}_g$ is unique up to a phase $\tilde{U}_g \to \varphi_g \tilde{U}_g \quad |\varphi_g| = 1$

the same equivalence class!!

$\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS $\longrightarrow$ unique equivalence class of proj. rep. of $\mathbb{Z}_2 \times \mathbb{Z}_2$

(iii) $\mathbb{Z}_2$ index $\sigma = \pm 1$

$$\tilde{U}_\alpha \tilde{U}_\beta = \sigma \tilde{U}_\beta \tilde{U}_\alpha \qquad \alpha, \beta \in \{x, y, z\}, \ \alpha \neq \beta$$

σ characterizes the transformation property of the MPS on the half-infinite chain

continuous modification of M^σ ($\sigma \in \{0, \pm\}$) that keeps the injectivity

↓

continuous change of $\tilde{U}_g$

↓

the index σ is invariant!

to change σ, one needs to break the injectivity

↓

gapless model.

- The index σ characterizes an SPT phase
- $\sigma = -1 \Rightarrow S_{LR} \geq \log 2$ PTBO

5.11 examples

case with U_x

$$\tilde{M}^s = \sum_{s'} \underbrace{\langle s | e^{-i\pi S^{(x)}} | s' \rangle}_{\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}} M^{s'}$$

$\therefore$ $\tilde{M}^+ = -M^-$, $\tilde{M}^0 = -M^0$, $\tilde{M}^- = -M^+$

we want to recover this by

$$\tilde{M}^s = e^{i\varphi_x} \tilde{U}_x^* M^s \tilde{U}_x \quad \text{for } s = 0, \pm 1$$

with some φ_x, $\tilde{U}_x$

- trivial state $M^0 = (1)$, $M^\pm = (0)$

  we take $\varphi_x = \pi$, $\tilde{U}_x = I$

- VBS state $M^+ = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}$, $M^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $M^- = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$

  we can take $\varphi_x = 0$, $\tilde{U}_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$

indices

- trivial state $\tilde{U}_x = I$, $\tilde{U}_z = I$ $\longrightarrow$ $\boxed{\sigma = 1}$
- VBS state $\tilde{U}_x = \underbrace{\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}}_{-i\sigma_x}$, $\tilde{U}_z = \underbrace{\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}}_{-i\sigma_z}$

$\tilde{U}_x \tilde{U}_z = -\tilde{U}_z \tilde{U}_x$ $\longrightarrow$ $\boxed{\sigma = -1}$ ← nontrivial SPT phase

$\boxed{\text{DISTINCT SPT PHASES!!}}$

⟨Perspective⟩

index of PTBO

well-defined for $\mathbb{Z}_2 \times \mathbb{Z}_2$ inv. injective MPS

provides a desired characterization of SPT phases.

extension to general models?

any unique gapped g.s. can be approximated by an MPS. Then we can use the PTBO theory!

↓

too optimistic.

- approximation theorems are not that precise
- the condition of injectivity is very strong
- it is likely that σ is determined only by D of an injective MPS

⋮

↓

We need a theory that is free from MPS scheme!

Ogata's index theorems DAY 2