⟨Preliminaries⟩

III C^*-algebra for the $S=1$ quantum spin system on the infinite chain $\mathbb{Z}$

the set of local operators

$\mathfrak{A}_{loc} := \{$ polynomials (with complex coefficents) of $S_j^{(\alpha)}$ with $j \in \mathbb{Z}$, $\alpha \in \{x, y, z\}$ $\}$

any $A \in \mathfrak{A}_{loc}$ depends on a finite number of spin operators.

C^*-algebra

$\mathfrak{A} = \overline{\mathfrak{A}_{loc}}$ ← completion with respect to the operator norm.

(Rem: in general a C^*-algebra is a Bnach-* algebra with $\|A^* A\| = \|A\|^2$ for any A.)

linear combinations ($\alpha A + \beta B$), conjugation (A^*) and norm ($\|A\|$) are def. as usual

III states on $\mathfrak{A}$

state ρ a linear map $\rho : \mathfrak{A} \to \mathbb{C}$

s.t. $\rho(1) = 1$

$\rho(A^* A) \geq 0$ for $\forall A \in \mathfrak{A}$

idea

$\rho(A)$ is the expectation value of A in the state ρ

note one always has $|\rho(A)| \leq \|A\|$

Rem. a useful property

Th. (Banach-Alaoglu) The set of all states on a C^*-algebra $\mathfrak{A}$ is compact w.r.t. the weak-$*$ topology

For a quantum spin system

$(\rho_j)_{j=1,2,\ldots}$ any infinite sequence of states.

$\exists$ a state ρ_∞, a subsequence $j(\ell)$ s.t. $j(\ell) < j(\ell+1)$

and $\rho_\infty(A) = \lim_{\ell \uparrow \infty} \rho_{j(\ell)}(A)$ for any $A \in \mathfrak{A}$.

▷ $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation on $\mathfrak{A}$.

$*$-automorphism

Def. $\Gamma : \mathfrak{A} \to \mathfrak{A}$ is a $*$-automorphism iff

(1) Γ is one to one

(2) Γ is linear

(3) $\Gamma(AB) = \Gamma(A)\Gamma(B)$ for $\forall A, B \in \mathfrak{A}$

(4) $\Gamma(A^*) = \Gamma(A)^*$ for $\forall A \in \mathfrak{A}$.

$*$-automorphism for $\mathbb{Z}_2 \times \mathbb{Z}_2$ Transformation

today I write $G \underset{\text{always}}{\underset{\varphi}{=}} \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, x, y, z\}$

define $*$-automorphism Γ_g for each $g \in G$ by

- $\Gamma_e = \mathrm{id}$
- for $\alpha, \beta \in \{x, y, z\}$

$$\Gamma_\alpha(S_j^{(\beta)}) = \begin{cases} S_j^{(\beta)} & \beta = \alpha \\ -S_j^{(\beta)} & \beta \neq \alpha \end{cases}$$

for $\forall j \in \mathbb{Z}$.

Clearly $\Gamma_g \circ \Gamma_h = \Gamma_{gh}$ for $\forall g, h \in G$.

III Hamiltonian and the ground states

general short-ranged Hamiltonian on $\mathbb{Z}$

- $\hat{H} = \sum_{j\in\mathbb{Z}} \hat{h}_j$ ← formal expression

$\hat{h}_j \in \mathfrak{A}_{loc}$: depends only on $S_j^{(\alpha)}$ with $|i-j| \leq R$ ← fixed range

$\|\hat{h}_j\| \leq h_0$ ← fixed

- $\hat{H}$ is ill-defined but

for $\forall A \in \mathfrak{A}_{loc}$, the commutator

$$[\hat{H}, \hat{A}] = [\sum_{j=-l}^{l} h_j, A] \in \mathfrak{A}_{loc}$$

for suff. large l

is well-defined.

ground states and gap

Def. A state ω is a g.s. iff

$\omega(A^*[H,A]) \geq 0$ for $\forall A \in \mathfrak{A}_{loc}$.

(for a finite system

$\langle GS| A^*[H,A]|GS\rangle = \langle GS|A^* H A|GS\rangle - E_{GS}\langle GS|A^*A|GS\rangle \geq 0$

$|\Psi\rangle = \dfrac{A|GS\rangle}{\|A|GS\rangle\|} \rightarrow \langle\Psi|H|\Psi\rangle \geq E_{GS}$)

Def. A unique g.s. ω is accompanied by a nonzero energy gap iff $\exists\,\gamma>0$ s.t.

$$\omega(A^*[H,A]) \geq \gamma\, \omega(A^*A)$$

for any $A \in \mathfrak{A}_{loc}$ s.t. $\omega(A)=0$.

(finite system (unique g.s. + gap)

$$\langle GS|A^* H A|GS\rangle \geq (E_{GS}+\gamma)\langle GS|A^*A|GS\rangle$$

$$\langle GS|A|GS\rangle = 0 \rightarrow \langle GS|\Psi\rangle = 0$$
)

☆ the notions of g.s. and energy gap can be defined only in terms of expectation values of local operators.

↓

physical!

III GNS construction → Gelfand-Naimark-Segal

Hilbert space for the theory?

the set $\bigotimes_{j\in\mathbb{Z}} \mathbb{C}^3$ is too large.

<u>Th.</u> Given a state ρ on $\mathfrak{A}$, one can construct

- a separable Hilbert space $\mathcal{H}_\rho$
- a representation π_ρ of $\mathfrak{A}$ on $\mathcal{H}_\rho$, i.e.

  linear map $\pi_\rho : \mathfrak{A} \to \overbrace{\mathcal{B}(\mathcal{H}_\rho)}$ → the set of all bounded operators on $\mathcal{H}_\rho$

  s.t. $\pi_\rho(AB) = \pi_\rho(A)\pi_\rho(B)$

  $\pi_\rho(A^*) = \pi_\rho(A)^*$ for $\forall A, B \in \mathfrak{A}$

- a vector $\Omega_\rho \in \mathcal{H}_\rho$ s.t.

  $\rho(A) = \langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle$ for $\forall A \in \mathfrak{A}$

  and $\{\pi_\rho(A)\Omega_\rho \mid A \in \mathfrak{A}\}$ is dense in $\mathcal{H}_\rho$.

$\boxed{(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho) : \text{GNS triple}}$

<u>rem.</u> • $\mathcal{H}_\rho$: "small" Hilbert space which consits of states that are "macroscopically the same" as ρ → physically natural (especially for QFT)

(• ρ is enough to recover the Hilbert space formalism.

• any state is written as a vector state)

idea of the construction

We have to construct a Hilbert space. How?

$\mathfrak{A}$ is alredy a linear space!

regard $A, B, \cdots \in \mathfrak{A}$ as "vectors"

with inner product

$$\langle A, B\rangle := \rho(A^* B)$$

def. of states

$$\text{then } \langle A, A\rangle = \rho(A^* A) \geq 0$$

$\langle\cdot,\cdot\rangle$ is not positive-definite.

equivalence relation

$$A \sim B \iff \langle A-B, A-B\rangle = 0$$

$$\mathcal{H}_\rho := \overline{\mathfrak{A}/\sim}$$

completion

set of equivalence classes

$\mathfrak{A}/\sim$ consists of $\Psi_A, \Psi_B, \ldots$

equiv. class including A

representation

$$\pi_\rho(A)\Psi_B = \Psi_{AB}$$

$$\Omega_\rho = \Psi_1$$

$$\langle \Omega_\rho, \pi_\rho(A)\Omega_\rho\rangle = \langle \Psi_1, \Psi_A\rangle = \rho(1A) = \rho(A)$$

⟨Ogata's index theory⟩

▷ **Assumptions** → Ogata's theory is much more general!!

- $S=1$ quantum spin system on $\mathbb{Z}$.
- $\hat{H} = \sum_{j\in\mathbb{Z}} h_j$ short ranged $G=\mathbb{Z}_2\times\mathbb{Z}_2$ inv. Ham.

$$\Gamma_g(h_j) = h_j \quad \text{for } \forall g\in G,\ \forall j\in\mathbb{Z}.$$

- ω a unique g.s. accompanied by a gap.

then ω is G-invariant.

i.e. $\omega(\Gamma_g(A)) = \omega(A)$ for $\forall g\in G$, $\forall A\in\mathfrak{A}$.

Matsui's theorem

examples · AKLT
· trivial

▷ **GNS rep. on the half-infinite chain**

$\mathfrak{A}_R$: C^*-algebra constructed from operators on the half-infinite chain $\{0,1,2,\cdots\}$

$\omega_R := \omega|_{\mathfrak{A}_R}$ restriction of ω onto $\mathfrak{A}_R$.

$$\mathfrak{A}_R,\ \omega_R \xrightarrow{\text{GNS const.}} (\mathcal{H}_R, \pi_R, \Omega_R)$$

the tailor-made Hilbert space that reflects the property of the g.s. on the half-infinite chain! HOPEFUL!

(iii) Representation of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathcal{H}_R$.

note. Γ_g is a $*$-automorphism on $\mathfrak{A}_R$.

for $g \in G$, define U_g by

$$U_g \Psi_A = \Psi_{\Gamma_g(A)} \quad \text{(and a natural extension to } \mathcal{H}_R\text{)}$$

$$\langle U_g \Psi_A, U_g \Psi_B \rangle = \langle \Psi_{\Gamma_g(A)}, \Psi_{\Gamma_g(B)} \rangle = \omega(\Gamma_g(A)^* \Gamma_g(B))$$

$$\overset{\text{G invariance of } \omega}{=} \omega(A^* B) = \langle \Psi_A, \Psi_B \rangle$$

$\therefore U_g$: unitary operator on $\mathcal{H}_R$. $\quad U_g^{-1}$ exists

since $\Gamma_g \circ \Gamma_h = \Gamma_{gh}$, $\quad U_g U_h = U_{gh}$

U_g $(g \in G)$ give a genuine rep. of G.

III $*$-automorphism on $\pi_R(\mathfrak{A}_R)$

we have ⊛ $U_g \pi_R(A) U_g^* = \pi_R(\Gamma_g(A))$ for $\forall g \in G$, $A \in \mathfrak{A}_R$.

$$\left(\begin{aligned} \because\ U_g \pi_R(A) \Psi_B &= U_g \Psi_{AB} = \Psi_{\Gamma_g(AB)} \\ &= \Psi_{\Gamma_g(A)\Gamma_g(B)} = \pi_R(\Gamma_g(A)) \Psi_{\Gamma_g(B)} \\ &= \pi_R(\Gamma_g(A)) U_g \Psi_B \end{aligned} \right)$$

Γ_g is a $*$-automorphism on $\mathfrak{A}_R$.

⊛ $\rightarrow$ $\pi_R(\mathfrak{A}_R)$ is closed under the action of $U_g(\cdot)U_g^*$

$$\hat{\Gamma}_g(X) := U_g X U_g^* \quad \text{for } X \in \pi_R(\mathfrak{A}_R)$$

$$\begin{cases} \hat{\Gamma}_g : *\text{-automorphism on } \pi_R(\mathfrak{A}_R) \\ \hat{\Gamma}_g \circ \hat{\Gamma}_h = \hat{\Gamma}_{gh} \quad \leftarrow \text{rep. of } G \\ \hat{\Gamma}_g(\pi_R(A)) = \pi_R(\Gamma_g(A)) \quad \text{for } \forall A \in \mathfrak{A}_R \leftarrow ⊛ \end{cases}$$

III Hilbert space $\tilde{\mathcal{H}}_R$

- commutant $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$

$$\mathfrak{M}' := \{A \in \mathcal{B}(\mathcal{H}) \mid [A,B]=0 \text{ for } {}^\forall B \in \mathfrak{M}\}$$

$$\mathfrak{M} \subset \mathfrak{M}'' = \mathfrak{M}'''' = \cdots$$

$$\mathfrak{M}' = \mathfrak{M}''' = \mathfrak{M}''''' = \cdots$$

recall

$$\Pi_R(\mathfrak{A}_R) \subset \mathcal{B}(\mathcal{H}_R)$$

$$\underbrace{\Pi_R(\mathfrak{A}_R)}_{C^*\text{-algebra}} \subset \underbrace{\Pi_R(\mathfrak{A}_R)''}_{\substack{\text{von Neumann algebra} \\ \text{(also a } C^*\text{-alg.)}}} \subset \mathcal{B}(\mathcal{H}_R)$$

$\Pi_R(\mathfrak{A}_R)''$ → closure of $\Pi_R(\mathfrak{A}_R)$ w.r.t. the weak topology

<u>essential fact</u>

ω is a unique gapped g.s.

$\downarrow$ Hastings

ω has area law entanglement

$\downarrow$ Matsui

ω is a pure split state

$\downarrow$

$\Pi_R(\mathfrak{A}_R)''$ is a type-I factor. — the most well-behaved class of von N. alg.

$\downarrow$ standard

$\exists$ a Hilbert space $\tilde{\mathcal{H}}_R$

$$\Pi_R(\mathfrak{A}_R)'' \cong \tilde{\mathcal{H}}_R$$

$\cong$ isomorphic

C*-algebra   Type-I factor   the set of all bounded operators

$$\pi_R(\mathfrak{A}_R) \subset \pi_R(\mathfrak{A}_R)'' \subset B(\mathcal{H}_R)$$

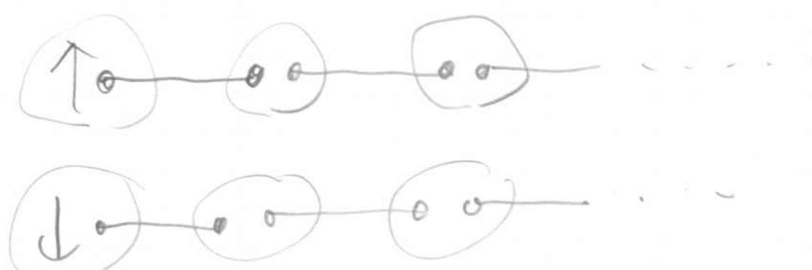
$$\pi_R(\mathfrak{A}_R)'' \cong B(\tilde{\mathcal{H}}_R)$$

$\mathcal{H}_R$ may be too large to describe the "edge states"

$B(\tilde{\mathcal{H}}_R)$: precise Hilbert space!

for AKLT..

$\tilde{\mathcal{H}}_R$



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other states

▭ projective rep. of G on $\tilde{\mathcal{H}}_R$ and the index

$\hat{\Gamma}_g$: $*$-automorphism on $\pi_R(\mathfrak{O}_R)$

extends to $\pi_R(\mathfrak{O}_R)''$ isomorphism $\pi_R(\mathfrak{O}_R)'' \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} B(\tilde{\mathcal{H}}_R)$

$\tilde{\Gamma}_g = \varphi \circ \hat{\Gamma}_g \circ \varphi^{-1}$ $*$-automorphism on $B(\tilde{\mathcal{H}}_R)$

of course $\tilde{\Gamma}_g \circ \tilde{\Gamma}_h = \tilde{\Gamma}_{gh}$ for $^\forall g, h \in G$

Th. (a varrant of Wigner's theorem)

Γ : a linear $*$-automorphism on $B(\mathcal{H})$.

$^\exists$ a untary U on $\mathcal{H}$ s.t.

$\Gamma(X) = UXU^*$ for $^\forall X \in B(\mathcal{H})$

thus

$^\exists \tilde{U}_g$: unitary on $\tilde{\mathcal{H}}_R$

s.t. $\tilde{\Gamma}_g(X) = \tilde{U}_g X \tilde{U}_g^*$ for $^\forall X \in B(\tilde{\mathcal{H}}_R)$

$\tilde{U}_g$ with $g \in G$ form a proj. rep. of G.

[iii] Ogata's $\mathbb{Z}_2$ index

a unique index $\sigma = \pm 1$ associated with $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant unique gapped g.s.

$$\tilde{U}_\alpha \tilde{U}_\beta = \sigma \tilde{U}_\beta \tilde{U}_\alpha \quad \text{for } \alpha, \beta \in \{x, y, z\}, \ \alpha \neq \beta.$$

<u>Th.</u> For a transl. invariant injective MPS,

$$\underset{\text{Ogata}}{\sigma} = \sigma_{\text{PTBO}}$$

<u>Th.</u> σ is invariant under any C^1-modification of $\mathbb{Z}_2 \times \mathbb{Z}_2$ inv. unique gapped g.s.

We have a well-defined $\mathbb{Z}_2$ index for any $\mathbb{Z}_2 \times \mathbb{Z}_2$ inv. unique gapped g.s.

$$\sigma_{\text{AKLT}} = -1, \quad \sigma_{\text{trivial}} = 1$$

<u>Cor.</u> the g.s. of the AKLT model and the trivial model cannot be connected by a C^1-path of $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant models satisfying Ogata's condition B.

▷ entanglement entropy in the nontrivial SPT phase

$\exists$ a density matrix (a trace-class positive op. with $\mathrm{Tr}_{\tilde{\mathcal{H}}_R}[\tilde{\rho}_R]=1$)
$\tilde{\rho}_R$ on $\tilde{\mathcal{H}}_R$ s.t.

$$\omega(A) = \mathrm{Tr}_{\tilde{\mathcal{H}}_R}[\tilde{\rho}_R\, \varphi(\pi_R(A))] \quad \text{for } \forall A \in \mathfrak{A}_R.$$

then

$$\omega(\Gamma_g(A)) = \mathrm{Tr}[\tilde{\rho}_R\, \varphi(\pi_R(\Gamma_g(A)))]$$

$$= \mathrm{Tr}[\tilde{\rho}_R\, \tilde{\Gamma}_g(\varphi(\pi_R(A)))]$$

$$= \mathrm{Tr}[\tilde{\rho}_R\, \tilde{U}_g\, \varphi(\pi_R(A))\tilde{U}_g^*]$$

$$= \mathrm{Tr}[\tilde{U}_g^*\, \tilde{\rho}_R\, \tilde{U}_g\, \varphi(\pi_R(A))]$$

$$\therefore \quad \tilde{\rho}_R = \tilde{U}_g^*\, \tilde{\rho}_R\, \tilde{U}_g \quad \text{for } \forall g \in G$$

exactly as yesterday if $\tilde{U}_\alpha \tilde{U}_\beta = -\tilde{U}_\beta U_\alpha$ $\quad \alpha \neq \beta,\ \alpha, \beta \in \{x, y, z\}$

then the nonzero e.v. of $\tilde{\rho}_R$ are even-fold deg.

Th. If $\sigma = -1$ then

$$S_{LR} := -\mathrm{Tr}_{\tilde{\mathcal{H}}_R}[\tilde{\rho}_R \log \tilde{\rho}_R] \geq \log 2.$$

a fully general version of the bound by PTBO!

⟨Perspective⟩

- Ogata's index theorems cover any quantum spin chain with a unique gapped g.s. with
  - any on-site symmetry — $\mathbb{Z}_2 \times \mathbb{Z}_2$, time-reversal, ⋮ → Ogata 2018
  - bond-centered inversion symmetry → Ogata 2019

- "topological" indices which are invariant under C^1-modification → "topological" phase transition

- lower bound for entanglement entropy

↓

physically satisfactory theories within mathematically "natural" setting

$$\pi_R(\mathfrak{A}_R) \subset \pi_R(\mathfrak{A}_R)'' \subset \mathcal{B}(\mathcal{H}_R)$$
$$\pi_R(\mathfrak{A}_R)'' \overset{?}{\cong} \mathcal{B}(\tilde{\mathcal{H}}_R)$$

REM. there are still few concrete examples of models with SPT order.

That the $S=1$ AF Heisenberg model has SPTO is still a conjecture! — "Haldane conjecture"

split states

ω is split

$\Updownarrow$

ω and $\omega|_{\mathfrak{A}_L} \otimes \omega|_{\mathfrak{A}_R}$ are quasi-equivalent

$\Updownarrow$

macroscopically the same

Two states on $\mathfrak{A}$

ρ_1 and ρ_2 are quasi-equivalent

$\Updownarrow$

(1) and (2) are valid

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(1) $\rho_1 \xrightarrow{GNS} (\mathcal{H}_1, \pi_1, \Omega_1)$

$\exists$ density matrix $\tilde{\rho}_2$ on $\mathcal{H}_1$ s.t.

$$\rho_2(A) = \mathrm{Tr}_{\mathcal{H}_1}[\tilde{\rho}_2\, \pi_1(A)] \quad \text{for any } A \in \mathfrak{A}$$

(2) The same with 1 and 2 switched.

commutant

$$\mathcal{M} \subset B(\mathcal{H})$$

$$\mathcal{M}' := \{ A \in B(\mathcal{H}) \mid [A, B] = 0 \text{ for } {}^{\forall} B \in \mathcal{M} \}$$

trivial property

$$\mathcal{M}_1 \subset \mathcal{M}_2 \Rightarrow \mathcal{M}_1' \supset \mathcal{M}_2' \quad \circledast$$

Since for ${}^{\forall}B \in \mathcal{M}'$ one has $[A, B] = 0$ for ${}^{\forall} A \in \mathcal{M}$

we see that

$$\underline{\mathcal{M} \subset \mathcal{M}''}$$

from $\circledast$ $\quad \mathcal{M}' \supset \mathcal{M}'''$

substitute $\mathcal{M}'$ into $\mathcal{M}$

$$\mathcal{M}' \subset \mathcal{M}'''$$

$$\mathcal{M}' = \mathcal{M}'''$$

$$\mathcal{M} \subset \mathcal{M}'' = \mathcal{M}'''' = \cdots$$

$$\mathcal{M}' = \mathcal{M}''' = \mathcal{M}''''' = \cdots$$