## (III) AKLT model and VBS state

- Why $|VBS\rangle$ is the exact g.s. of $H_{AKLT}$ ?

$$H_{AKLT} = \sum_{j=1}^{L} \left\{ \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \frac{1}{3} (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2 \right\}$$

$$= \sum_{j=1}^{L} \left\{ 2 \underbrace{P_2 [\mathbf{S}_j + \mathbf{S}_{j+1}]} - \frac{2}{3} \right\}$$

$\hookrightarrow$ projection on the space with

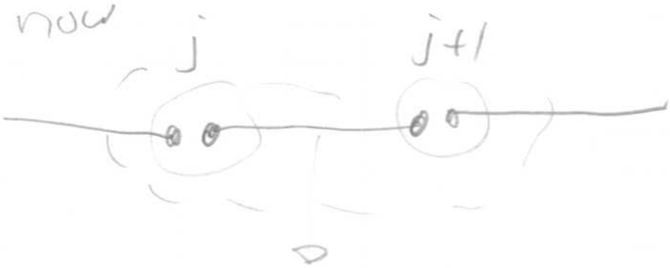
$$(\mathbf{S}_j + \mathbf{S}_{j+1})^2 = 2(2+1)$$

the e.v. of $(\mathbf{S}_j + \mathbf{S}_{j+1})^2 \Rightarrow S_{tot}(S_{tot}+1)$

with $S_{tot} = 0, 1, 2$

here.

now




~~We~~

only Two $S=1/2$'s

$\underline{S_{tot} = 0 \text{ or } 1}$ ,

$\therefore$

$$\underline{P_2[\ ]}\,|VBS\rangle = 0$$

the min. e.v. of $P_2[\ ]$.

- AKLT model on an open chain

$$H^{\text{open}}_{\text{AKLT}} = \sum_{j=1}^{L-1} \left\{ S_j \cdot S_{j+1} + \frac{1}{3}(S_j \cdot S_{j+1})^2 \right\}$$

$$= \sum_{j=1}^{L-1} \left\{ 2 P_2(S_j + S_{j+1}) - \frac{2}{3} \right\}$$

then any state


are g.s.

4-fold deg.

edge state!

$L \nearrow \infty$

$$\langle \mathrm{VBS}^{\uparrow} | S_j^{(z)} | \mathrm{VBS}^{\uparrow} \rangle = -2(-3)^{-j}$$

exponentially localized near the edge

M-4

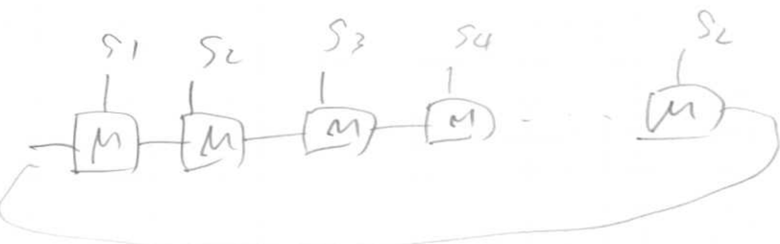
## ⟨More about MPS⟩

$M^s_{\alpha\alpha'}$


$$M^s = \left(M^s_{\alpha,\alpha'}\right)_{\alpha,\alpha'=1,\dots,D}$$

$$|\Phi\rangle = \sum_{\mathbf{s}} \mathrm{Tr}\left[M^{s_1}\cdots M^{s_L}\right]|\mathbf{s}\rangle$$

$$= \sum_{\mathbf{s}} \sum_{\alpha_1,\dots,\alpha_L=1}^{D} M^{s_1}_{\alpha_1\alpha_2} M^{s_2}_{\alpha_2\alpha_3}\cdots M^{s_L}_{\alpha_L\alpha_1}|\mathbf{s}\rangle$$




norm

$$\langle\Phi|\Phi\rangle = \sum_{\mathbf{s}} \overline{\mathrm{Tr}\left[M^{s_1}\cdots M^{s_L}\right]}\,\mathrm{Tr}\left[M^{s_1}\cdots M^{s_L}\right]$$

$$= \sum_{\mathbf{s}} \sum_{\alpha_1,\dots,\alpha_L} \overline{M^{s_1}_{\alpha_1\alpha_2}}\cdots\overline{M^{s_L}_{\alpha_L\alpha_1}} \sum_{\beta_1\cdots\beta_L} M^{s_1}_{\beta_1\beta_2}\cdots M^{s_L}_{\beta_L\beta_1}$$

$$= \sum_{\substack{\alpha_1,\dots,\alpha_L\\ \beta_1\cdots\beta_L}} \left(\sum_s \overline{M^s_{\alpha_1\alpha_2}} M^s_{\beta_1\beta_2}\right)\left(\sum_s \overline{M^s_{\alpha_2\alpha_3}} M^s_{\beta_2\beta_3}\right)\cdots$$

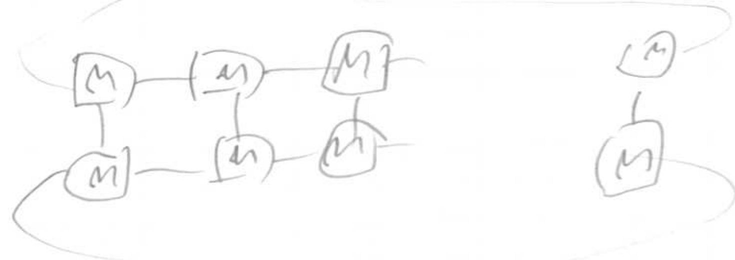
$$\cdots\left(\sum_s \overline{M^s_{\alpha_L\alpha_1}}\, M^s_{\beta_L\beta_1}\right)$$

$$= \mathrm{Tr}\left[\mathcal{M}^L\right]$$

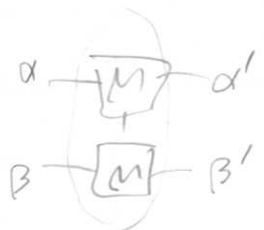
$\mathcal{M}$ $D^2 \times D^2$ matrix : transfer matrix

$$\mathcal{M}_{(\alpha,\beta),(\alpha',\beta')} = \sum_s \overline{M^s_{\alpha\alpha'}} M^s_{\beta\beta'}$$

$\langle \bar{\Phi} | \Phi \rangle =$



$\mathcal{M} =$



you can compute
expec. values

another way of stating the injectivity

$|\Phi\rangle$ is injective iff

(i) $\sum_{s=-S}^{S} M^s (M^s)^* = \lambda I \qquad \lambda > 0$

(ii) ~~$\mathcal{M}$ is~~ $\lambda$ is the nondeg. e.v. of $\mathcal{M}$
with the largest absol. values

# III) group cohomology

$G$: a group, $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$

n-cochain $\omega$ is a map

~~map~~ $\omega : \underbrace{G \times \cdots \times G}_{n} \to U(1)$

$C^n(G, U(1))$ the set of all n-cochains → abelian group

coboundary homomorphism

$d : C^n(G, U(1)) \to C^{n+1}(G, U(1))$

$$d\omega(g_1, \ldots, g_{n+1}) = \omega(g_2, \ldots, g_{n+1}) \left(\omega(g_1, \ldots, g_n)\right)^{(-1)^{n+1}}$$
$$\times \prod_{i=1}^{n} \left(\omega(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1})\right)^{(-1)^i}$$

$\omega \in C^2(G, U(1))$

$$d\omega(g_1, g_2, g_3) = \frac{\omega(g_2, g_3)\ \omega(g_1, g_2 g_3)}{\omega(g_1, g_2)\ \omega(g_1 g_2, g_3)}$$

$\omega \in C^3(G, U(1))$

$$d\omega(g_1, g_2, g_3, g_4) = \frac{\omega(g_2, g_3, g_4)\,\omega(g_1, g_2 g_3, g_4)\,\omega(g_1, g_2, g_3)}{\omega(g_1 g_2, g_3, g_4)\,\omega(g_1, g_2, g_3 g_4)}$$

it is found in general that

$d \circ d\omega = 1$ for $\forall \omega \in C^n(G, U(1))$

$\omega \in C^1(G, U(1))$

$$d\omega(g_1, g_2) = \frac{\omega(g_2)\,\omega(g_1)}{\omega(g_1 g_2)}$$

the set of n-cocycles

$$Z^n(G, U(1)) := \{\omega \in C^n(G, U(1)) \mid d\omega = 1\}$$

the set of n-coboundaries

$$B^n(G, U(1)) := \{\omega \in C^n(G, U(1)) \mid {}^\exists \tilde{\omega} \in C^{n-1}(G, U(1)) \text{ s.t. } \omega = d\tilde{\omega}\}$$

since $d \circ d\, \omega = 1$, $B^n(G, U(1)) \subset Z^n(G, U(1))$

n-th group cohomology

$$H^n(G, U(1)) = Z^n(G, U(1)) / B^n(G, U(1))$$

$\downarrow$

equivalence classes of n-cochains.

$$\omega \sim \omega' \iff {}^\exists \tilde{\omega} \text{ s.t. } \frac{\omega}{\omega'} = d\tilde{\omega}$$

projective rep of $G$ and $H^2$

$\omega(g, h) \in U(1)$

$$U_g U_h = \omega(g, h) U_{gh}.$$

associativity

$$U_{g_1}(U_{g_2} U_{g_3}) = (U_{g_1} U_{g_2}) U_{g_3}$$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$$\underbrace{\omega(g_2, g_3)\, \omega(g_1, g_2 g_3)}\, U_{g_1 g_2 g_3} = \underbrace{\omega(g_1, g_2)\, \omega(g_1 g_2, g_3)}\, U_{g_1 g_2 g_3}$$

the same

$\omega \in Z^2(G, U(1))$

equivalent proj. rep. $U'_g = \nu(g) U_g$ $\quad \nu(g) \in U(1)$

$$U'_g U'_h = \nu(g)\nu(h) U_g U_h = \nu(g)\nu(h)\,\omega(g,h)\, U_{gh}$$

$$= \underbrace{\frac{\nu(g)\nu(h)}{\nu(gh)}\omega(g,h)}\, U'_{gh}$$

if we write $U'_g U'_h = \omega'(g,h)\, U'_{gh}$

$$\frac{\omega'(g,h)}{\omega(g,h)} = \frac{\nu(g)\nu(h)}{\nu(gh)} = d\nu(g,h)$$

thus $\omega' \overset{eq.}{\sim} \omega \iff (U'_g) \overset{eq.}{\sim} (U_g)$

$H^2(G, U(1))$: eq. classes of proj. rep.

"physical" object related to $H^3$

suppose we have a "quantized" rel.

$$\tilde{U}_g \tilde{U}_h = \hat{\Omega}(g,h) \tilde{U}_{gh}$$

with some unit. op. s.t. $U_f \hat{\Omega}(g,h) U_f^*$ is also a unitary

for some $\hat{\Omega}(g,h)$, we get

$$(\tilde{U}_f \tilde{U}_g)\tilde{U}_h = \underbrace{\omega(f,g,h)}_{\in U(1)} \tilde{U}_f(\tilde{U}_g \tilde{U}_h)$$

violation of associativity!

$U_g$ are NOT unitary operator?

consistency ~~rel.~~ shows that

$$\omega \in Z^3(G, U(1))$$ !

~~$\omega \sim \omega'$~~

# ⟨SPT order in higher dimensions⟩

Chen, Gu, Liu, Wen 2012, 2013. (Science, PRB)

STP phases in $d$-dim $\leftrightarrow H^{d+1}(G, U(1))$

with symmetry group

(incomplete classification in $d \geq 1$. cobordism ?)

## (1.1) The simplest example. → simpler than the CZX model

Chen, Gu, Liu, Wen 2013

Miller-Miyake 2016, Yoshida 2016.

finite group $G$ $\omega \in Z^3(G, U(1))$ ~~$G\times G$~~

(example $G = \mathbb{Z}_2$, ~~$\mathbb{Z}\ \mathbb{Z}\ H^3(G, U$~~

$H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$ ← two phases!

trivial $\omega(f,g,h) = 1$ for all $f, g, h$

nontrivial $\omega(f,g,h) = \begin{cases} -1 & (f,g,h) = (-,-,-) \\ 1 & \text{otherwise} \end{cases}$ )

~~weight~~

phase factor

$$\Psi(a,b,c) := \omega(c^{-1}b, b^{-1}a, a^{-1}) \qquad a, b, c \in G$$

↑ not a 3-cocycle

(nontrivial $\Psi(a,b,c) = \begin{cases} -1 & (a,b,c) = (-,+,-) \\ 1 & \text{otherwise} \end{cases}$ )

transformation rule

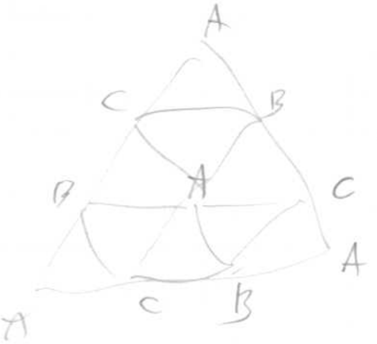
from the cocycle condition.

$$\psi(g^{-1}a, g^{-1}b, g^{-1}c) = \psi(a,b,c)\,\frac{\omega(g^{-1},a,ab)\,\omega(g^{-1}a,b,bc)}{\omega(g^{-1},a,ac)}$$

$a, b, c, g \in G$

~~state~~

lattice $\Lambda$: triangular lattice without ~~with or without~~ boundaries.




$\Lambda = A \cup B \cup C$

$t$: triangle


$a(t),\ b(t),\ c(t)$ sites on $t$

$\ \ \in \qquad \in \qquad \in$

$\ \ A \qquad B \qquad C$

$\mathcal{J}_\Delta$ : the set of ~~upright~~ upward triangles

$\mathcal{J}_\nabla$ " downward "

"spin" system

state on a single site $|s\rangle$ $\quad s \in G$.

~~acting~~ ~~on~~ on the whole lattice $|\mathbf{s}\rangle = \bigotimes_{x\in\Lambda} |s_x\rangle_x$ $\quad s_x \in G$

$\mathbf{s} = (s_x)_{x\in\Lambda}$

def

$U_g|\mathbf{s}\rangle = |g\mathbf{s}\rangle$ $\quad g\mathbf{s} = (gs_x)_{x\in\Lambda}$.

## The state

phase factor
$$\Phi(\mathbb{S}) = \left(\prod_{t\in J_\Delta} \psi(s_{a(t)}, s_{b(t)}, s_{c(t)})\right)$$
$$\times\left(\prod_{t\in J_\nabla} \psi(s_{a(t)}, s_{b(t)}, s_{c(t)})\right)^{-1}$$

$$|\Phi\rangle = \sum_{\mathbb{S}} \underbrace{\Phi(\mathbb{S})}_{\text{only the phase is modulated}} |\mathbb{S}\rangle$$

$|\Phi\rangle$ has zero correlation length.

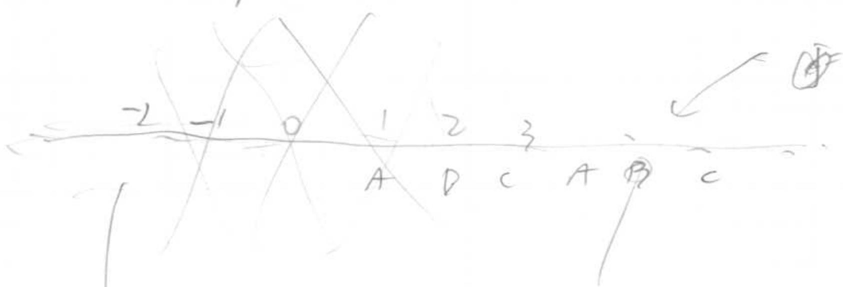
$$U_g|\Phi\rangle = \sum_{\mathbb{S}} \Phi(\mathbb{S})|g\mathbb{S}\rangle$$
$$= \sum_{\mathbb{S}} \underbrace{\Phi(g^{-1}(\mathbb{S}))}_{=\,\Phi(\mathbb{S})} |\mathbb{S}\rangle$$


$$= |\Phi\rangle$$ G-invariant.

↓

but in a nontrivial manner

formally consider $|\Psi\rangle$ on the infinite triangle lattice




fictitiously "cut" the whole lattice by a line

$j \in \mathbb{Z}$
the sites on the line

the transformation of the lower half.

$$U_g|\Psi_{half}\rangle = \sum_{\mathbb{S}} \underbrace{\Psi_{half}(g^{-1}\mathbb{S})}\,|\mathbb{S}\rangle$$

$$= \sum_{\mathbb{S}} \underbrace{\left(\prod_j \xi_j(g; S_j, S_{j+1})\right)} \Psi_{hal}(\mathbb{S})\,|S\rangle$$

$$\xi_j(g;S,S') = \begin{cases} \omega(g^{-1}, S, SS') & j \in A \cup B \\ \dfrac{1}{\omega(g^{-1}, S', S'S)} & j \in C \end{cases}$$

then

$$U_g U_h |\Psi_{half}\rangle = \sum_{\$}{}' \left( \prod_j \underbrace{\xi_j(h; g^{-1}s_j, g^{-1}s_{j+1}) \xi_j(g; s_j, s_{j+1})}_{\odot} \right) \Psi_{half}(\$) |\$\rangle$$

and

$$U_{gh} |\Psi_{half}\rangle = \sum_{\$}{}' \left( \prod_j \underbrace{\xi_j(gh; s_j, s_{j+1})}_{\triangle} \right) \Psi_{half}(\$) |\$\rangle$$

from the 3-cocycle condit.

$$\underbrace{\xi_j(h; g^{-1}s_j, g^{-1}s_{j+1}) \xi_j(g; s_j, s_{j+1})}_{\odot}$$

$$= \frac{1}{\omega(h^{-1}, g^{-1}, s_j)} \underbrace{\xi_j(gh; s_j, s_{j+1})}_{\triangle} \omega(h^{-1}, g^{-1}, s_{j+1})$$

So $U_g U_h |\Psi_{half}\rangle = U_{gh} |\Psi_{half}\rangle$

genuine rep.

further ~~ca~~ "cut" the line into two

$$U_g U_h |\Psi_{\text{quantum}}\rangle = \sum_{\mathbf{s}} \left( \prod_{j=1}^{\infty} \xi_j(h; g^{-1}s_j, g^{-1}s_{j+1})\, \xi_j(g; s_j, s_{j+1}) \right) \Psi_q(\mathbf{s})\, |\mathbf{s}\rangle$$

$$\cancel{U_g U_h |\Psi_q\rangle = \sum_{\mathbf{s}} \prod_{j=1}^{\infty} \xi_j(gh}$$

$$= \sum_{\mathbf{s}} \frac{1}{\omega(h^{-1}, g^{-1}, s_1)} \prod_{j=1}^{\infty} \xi_j(gh; s_j, s_{j+1})\, \Psi_q(\mathbf{s})\, |\mathbf{s}\rangle$$

$$= \hat{\Omega}(g,h)\, U_{gh} |\Psi_{qu}\rangle$$

$$\hat{\Omega}(g,h) = \sum_{s} \frac{1}{\omega(h^{-1}, g^{-1}, s)} |s\rangle_1 {}_1\langle s|$$

unitary op. ~~acting~~
on site 1.

$$U_g U_h = \Omega(g,h)\, U_{gh}$$

"quantized" proj. rep.

$$(U_f U_g) U_h = \Omega(f,g) U_{fg} U_h = \Omega(f,g)\, \Omega(fg,h)\, U_{fgh}$$

$$U_f (U_g U_h) = U_f\, \Omega(g,h)\, U_{gh} = U_f\, \Omega(g,h)\, U_f^{*}\, U_f U_{gh}$$

$$= (U_f\, \Omega(g,h)\, U_f^{*})\, \Omega(f,gh)$$

$$\downarrow$$

$$(U_f U_g) U_h = \omega(f,g,h)\, U_f (U_g U_h)$$

associativity is violated!!

NO MATH yet.

~~Molnar~~

similar analysis

Chen, Liu, Wen 2011 PRB

Chen, Gu, Liu, Wen 2013 PRB

) CZX model

difficulty ↓

Molnar, Ge, Schuch, Cirac 2018 preprint