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References for the lecture series "Symmetry protected topological phases, topological indices, and operator algebra in quantum spin chains"

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Here I will list all the references that will be mentioned in my lectures. Please note that this is not the list of recommended reading; most of these references are original publications, which are urually not easy to fully comprehend.

As far as I know the best account on the subject is my book [35], which will be published next year. If you read Japanese my (very old) review [33] may be useful in learning the background. (But the review was written much before the invention of the notion of SPT.)

Note that all the links from the reference list are available to everyone.

Day 1 Haldane's seminal works in 1983 are [16, 17]. He indeed had an earlier paper [15] with essentially the same conclusion, but the paper was not published. (According to Haldane one of the referees stated that it was "in manifest contradiction to fundamental principles of physics" [18].)

The AKLT model was introduced in 1987 in [1, 2]. The existence of a "topological" phase transition between the AKLT model and the trivial model was proved for the first time in 2018 in [34]. For much more standard notion of symmetry breaking, see, e.g., the recent review [3].

It was argued by Chen, Gu, and Wen [8] that all one-dimensional spin models with a unique gapped ground state can be connected continuously. See also [9, 32]. This claim has been rigorously established within matrix product states by Ogata in [24, 25, 26].

Gu and Wen [14] proposed the notion of SPT 1n 2009, and argued that the the AKLT model belongs to a nontrivial SPT phase. Pollmann, Turner, Berg, and Oshikawa [30, 31] determined the class of symmetry necessary for the "protection", and also developed their index theory within matrix product states. Ogata's rigorous index theorems, which are the topic of the second day, are developed in [27, 28].

The formalism of MPS was discovered in 1989 by Fannes, Nachtergaele, and Werner [11, 12, 13], who also developed a complete general theory of MPS.

Before the index theory of Pollmann, Turner, Berg, and Oshikawa's [30, 31], there was a work by Perez-Garcia, Wolf, Sanz, Verstraete, and Cirac [29], where essential techniques and a similar idea were developed. Back in 2001, Matsui developed a mathematical theory for quantum spin chains based on projective representations of group symmetry, not only for MPS but for more general pure states which satisfy the split property [21].

Day2 Ogata's rigorous index theorems are developed in [27, 28]. Bratteli and Robinson's textbook [4, 5] is a definitive reference on the operator algebraic approach to quantum many-body systems.

That the AKLT model on the infinite chain has a unique gapped ground state was proved by Matsui in 1997 [20].

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The phase diagrams of the anisotropic model A schematic phase diagram of the $S = 1$ chain can be found, e.g., in Fig. 1.1 of [19]. The amazing phase diagram for the $S = 2$ chain can be found in Fig. 3 of [36].

SPT in higher dimensions Chen, Gu, Liu, and Wen [6, 7] argued that SPT phases in d-dimensional quantum spin systems are classified in terms of the group cohomology $\mathrm{H}^{d+1}(G, \mathrm{U}(1))$ of the symmetry group G.

The first example of a model in a genuine two-dimensional SPT phase is the CZX model constructed by Chen, Liu, and Wen [10]. The simple model on the triangular lattice was (briefly) introduced by Chen, Gu, Liu, and Wen in section II.F of [7]. Closely related models were studied in detail by Miller and Miyake [22] and Yoshida [37].

A direct characterization of the SPT order in terms of 3-cocycle was discussed for the CZX in [10] and for a more general class of models in [7]. A more systematic and general analysis was made recently in [23].

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