

Instruction Methods for Solving Word Problems in Mathematics Education

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ABSTRACT

The purpose of this paper is to report on three methodologies that I practise in business mathematics lectures. In business mathematics, a problem to find is more important than a problem to prove. Thus, my target is a problem to find. The first methodology is to solve a problem visually to the extent possible, not only using existing algebraic methods but also a visual method that includes mathematical software, such as Maple and Mathematica. In business mathematics today, problem solving using the visual approach is more important than that using the algebraic approach because business requires the speedy modeling of the target and prompt analysis of the model. Thus we can say that the visual solution skill is also required as digital literacy in the business mathematics field. Therefore, lecturers are recommended to take greater advantage of this new visual approach. The second methodology is to practise Polya's famous heuristics, described in his book, "How to Solve It". It is particularly important for students to first write down the given data and the unknown to understand a word problem. The third methodology is to have students construct rule databases in their heads to make a deduction. Using deduction, they can easily find the connection between the given data and the unknown to devise a solution plan. Based on my experience with deductive database system construction, I think that the procedures used in a deductive system can be applied to students' solution procedures. In my lectures, I repeatedly teach students deductive patterns. Thinking patterns using these three methodologies are helpful in cultivating students' ability to solve future problems by themselves.

1. Introduction

The research presented here is on instruction methods for solving word problems in business mathematics. In the faculty of Economics at my university, one topic that I lecture on is business mathematics. In business mathematics, a problem to find is more important than a problem to prove. Thus, my target is a problem to find.

Computer software for mathematics has become widely diffused in recent years. Software such as Maple^{*} and Mathematica[†] are well known. For example, Maple can be used on every PC on our cam-

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* Maple Web site: <http://www.maplesoft.com/products/maple/> (accessed on 2008/11/02)

† Mathematica Web site: <http://www.wolfram.com/products/mathematica/index.html> (accessed on 2008/11/02)

pus. Many mathematics textbooks come with CD-ROMs that contain either graphing calculator or computer functions (Strauss, Bradley et al. 2002). However, few lecturers have introduced computer software in class. Cost is the biggest reason. The need for an IT room, including several instructors, makes the cost much higher than for an ordinary lecture. Another significant factor is the worry of technical difficulties arising on both the lecture and student sides.

Even if devices and instructors are arranged, the problem of instruction method remains. The worst-case scenario in such computer-based instruction is that the lecturer focuses more on operating the computer than on thinking about the mathematics problems. We must keep in mind that our mission is to cultivate student ability to solve future problems by themselves. It is recommended that lecturers maintain a balance between computer operations and problem-solving methods in class.

I am a computer scientist and have developed several e-Learning systems for business mathematics (Shirota 2004; Shirota 2004; Shirota 2004; Shirota 2005; Shirota 2006). I have also developed a deductive system for economic mathematics (Shirota 2006). Using these e-Learning systems, I currently teach business mathematics to several classes. Two classes are big, so students can only see my visual teaching materials. In each of the other three classes, however, there are fewer than 15 students. There, students can learn business mathematics through visualization, using computers and Maple.

The first methodology is visualization using computers. The second one is practising Polya's heuristics, which are described in "How to Solve It" and are well known for solving problems (Polya 2004). It is particularly effective for students to first write down the given data and the unknown to understand the word problem. The third methodology is to have students simulate the deductive database system in their heads. First, the instructor tells students to make two kinds of rule databases in their heads. Students then make the deduction by applying the rules in the databases. Using deduction, they can easily find the connection between the given data and the unknown to devise a solution plan. This third methodology may be a tiny improvement based on Polya's heuristics, derived from my experience in deductive database construction. The goal of this method is to make it a practice for students to use deduction very consciously and in a focused manner. I call this the "inference engine simulating method".

In the following section, the three instruction methodologies that I practise are briefly explained. In Section 3, as in the first, I will explain my visual approach using Maple. In Sections 4, Polya's heuristics and the deduction method that I practise are presented. Finally I comment on the impact of these instruction methodologies and conclude the paper.

2. Three Methodologies that I Practise

In this section, I will briefly present three methodologies that I practise in my business mathematics lectures. They are as follows:

- (1) Visual approach, using Maple,
- (2) Polya's method described in "How to Solve It", and
- (3) The application of deduction to rule databases.

Following are some questions and commands that I use frequently in the respective phases:

- (1a) Solve the same problem both visually and algebraically, if possible.
- (1b) Grasp the whole image of the function visually.

- (2a) What is the unknown?
- (2b) What is the given data?
- (2c) What are the conditions?
- (3a) Construct rule databases in your head.
- (3b) Find the relationship between the unknown and the given data.
- (3c) Search the databases in your head with the given keyword.
- (3d) Does that rule exist in the economics rule database, or the mathematics rule database?
- (3e) Memorize a rule correctly when you put it into your rule database, paying special attention to its adaptation condition.
- (3f) Annotate the rule with many keywords.

In the following sections, I explain details of the methodologies.

2. Visual Solution Methods using Maple

In this section, I explain the visual solution methods that I practise using Maple.

There are two methods for solving a mathematics problem: (A) the existing algebraic method, and (B) the new visual approach. In general, it is expensive to sketch a graph of a function. Therefore, lecturers tend mainly to teach the algebraic solution method. However, the recent increase in computational power has made it possible to visualize complicated mathematical expressions. This means that users can easily draw a 3-D graph and contour map and dynamically change the viewpoint from which they observe the graph (See Figure 1).

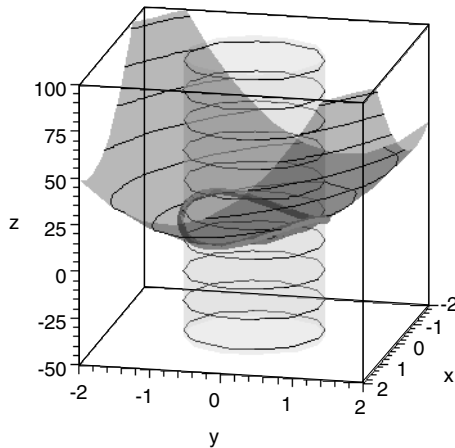


Figure 1. A 3-D contour map of a principal component analysis drawn by Maple.

I think that in business mathematics today, problem solving using the visual approach is more important than that using the algebraic approach. Business requires the speedy modeling of the target and prompt analysis of the model. Of course, students must definitely acquire the skills to solve problems al-

gebraically. However, they should first grasp the whole image of a given mathematical expression through visualization. Using various types of computer graphics, students solve the problem visually as the first step, if possible (sometimes a problem cannot be solved using the visual approach). Next, it is necessary to separate the problem into parts and write them down. Knowledge of graphic images can also help students solve the problem algebraically. Because the graphics are very impressive, they help students identify the direction and goal of the algebraic solution process.

The biggest deficit of the visual approach is that it is impossible to draw graphs with more than three dimensions. In other words, the multivariable function of more than two independent variables cannot be drawn. (A decrease in dimension would be needed.) Despite some restrictions in the visualization, the visual approach is more effective than the algebraic solution process for intuitively understanding the given functions.

Optimization problems are the most suitable mathematical problems for visual solution because we can identify the local maximum/minimum point instantly from the graphics. Thus, I will provide a constrained optimization problem (Jaques 2003) as a typical example of visualization. The other example that I will show is the curve shift effect of the national income determination problem (Jaques 2003).

2.1 Constrained Optimization Problem

In business mathematics, optimization problems are the most important. They include the profit maximizing problem and the cost minimizing problem.

Now let us consider the following problem:

The cost function is given by

$$C = 12x^2 - 6xy + 6y^2$$

where x and y denote the number of two kinds of goods purchased. The restriction is $x+y=72$. Find the optimal value of the cost.

The 3-D image of the visualization is illustrated in Figure 2. The technical point is the intersection function of Maple, which calculates the intersection part and marks it graphically, as shown in Figure 3.

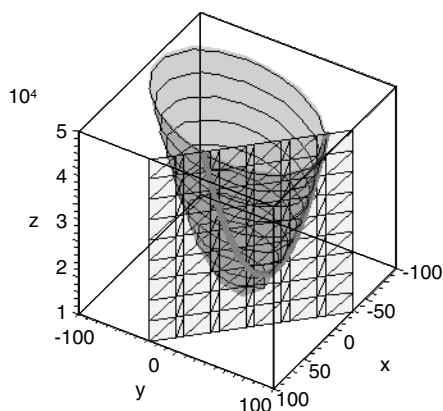


Figure 2. The cost minimizing problem graphs under the condition $x+y=72$.

In my class, students can draw these graphs and change the viewpoint dynamically to find the local maximum/minimum points on their individual computers. Then they can make the 2-D image on the x-z plane or the y-z plane by moving the graphs so that they can obtain the unknown x and y values. This is a typical example of my visual solution method using Maple.

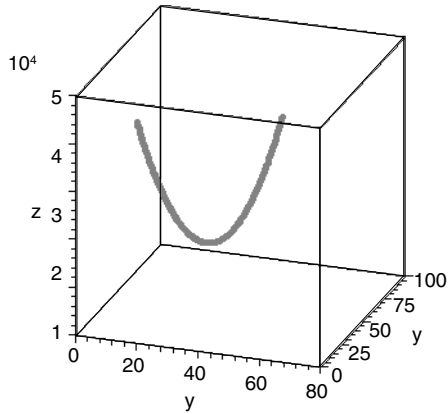


Figure 3. The intersection part of the original function and the condition in Figure 2.

The intersection function helps us visually solve a set of simultaneous equations with three variables. If there is a solution, the solution point or line is displayed graphically as shown in Figure 4, where the set of equations are given as follows:

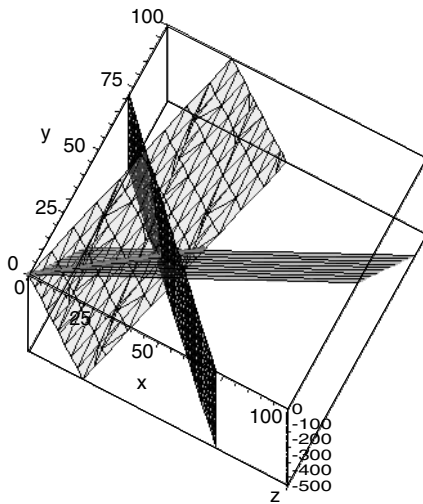


Figure 4. Visual solving solution method for the set of simultaneous equations.

If it were not for Maple and similar software, it would be too difficult to sketch the 3-D planes as shown here, although my students now take the graphics for granted and utilize the visual approach.

2.2 Curve Shift Effect on the National Income Determination Problem

In economics mathematics, there are many shift effect problems. In this section, let us consider the analysis of IS and LM schedules on the national income determination problem. In the determination problem, any increase in money supply (M_s) increases the intercept, and the LM curve shifts upwards on the R-Y plane where R represents the interest rate and Y represents the national income. The problem is asked as follows: “What is the effect of an increase in money supply (M_s) on the equilibrium level of national income Y”? Figure 5 and 6 illustrate the problem to be solved.

This teaching material was generated by e-Math, a system that I developed. The e-Math system automates the XML file from the input specification file and then displays the XML file through the XLST file, which is written in advance.

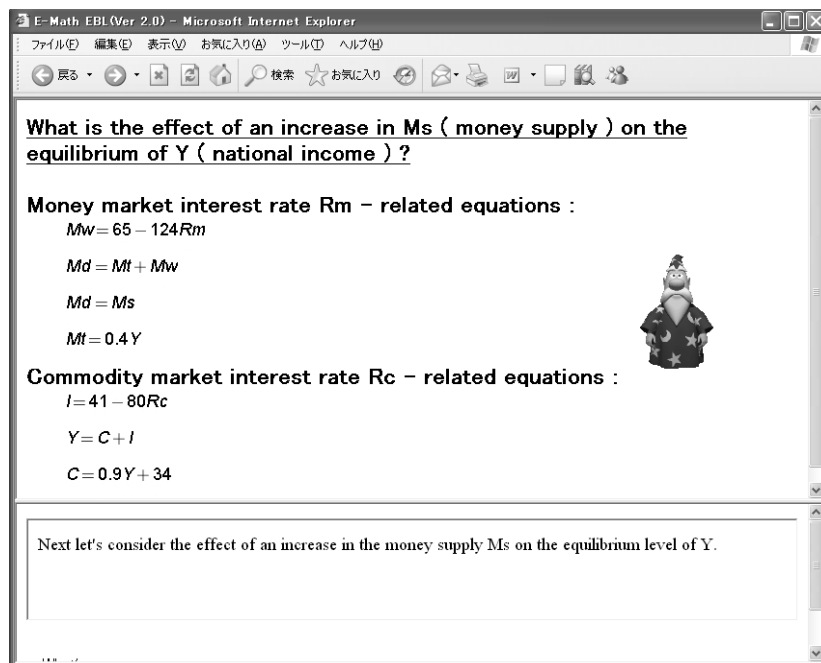


Figure 5. The screen of teaching materials developed to explain the shift problem.

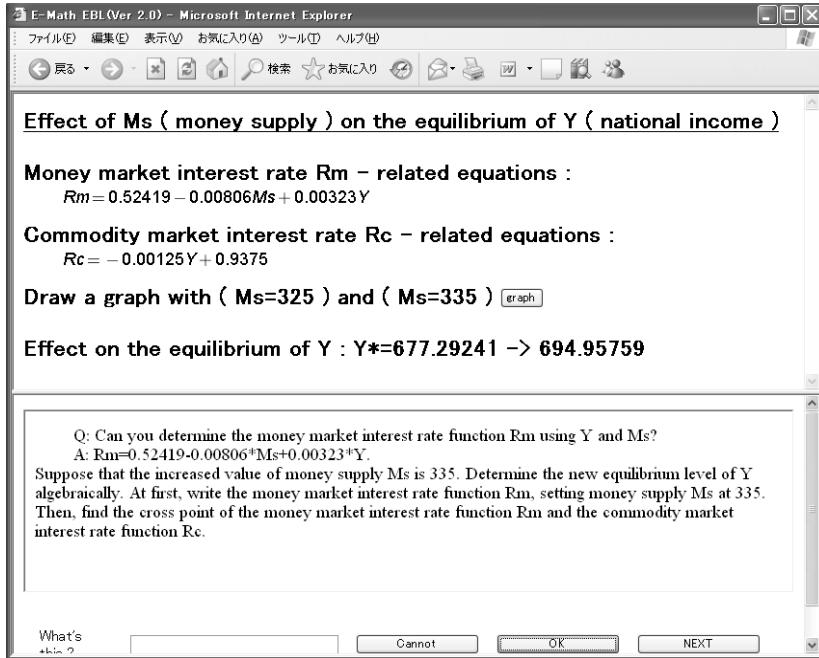


Figure 6. The screen of teaching materials developed to explain the shift problem.

In this teaching material, the Maple script, Maplet, is attached. The graphs generated by Maplet are shown in Figures 7 to 9. In Figure 7, the two contour lines correspond to the conditions $Ms=325$ and $Ms=335$. Students can shift the viewpoints interactively to map the contour lines to a 2-D plane of Y - R_m (money market interest). The shift effects of the money supply on the unknown “national income Y ” can be sketched, as shown in Figures 8 and 9.

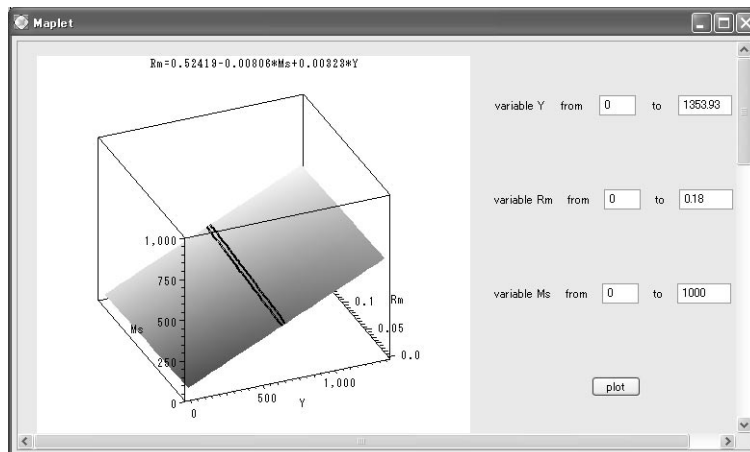


Figure 7. The 3-D image of two LM curves corresponding to the two values of money supply (Ms). Students can move the 3-D image dynamically on their PCs.

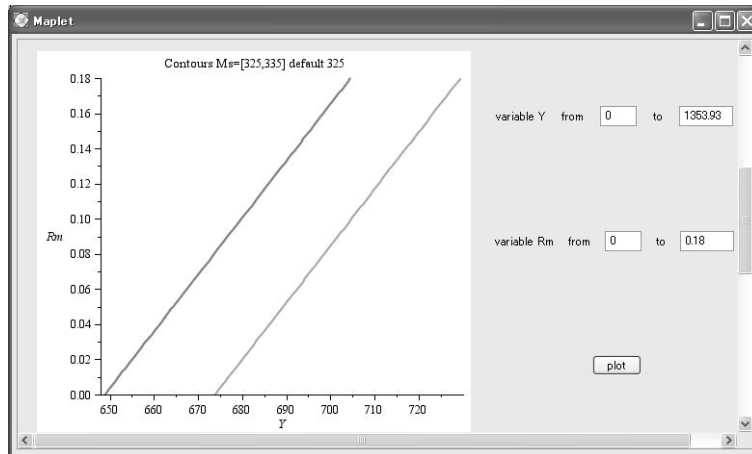


Figure 8. The 2-D image of the two LM curves of Figure 6. The curve shifts to the right.

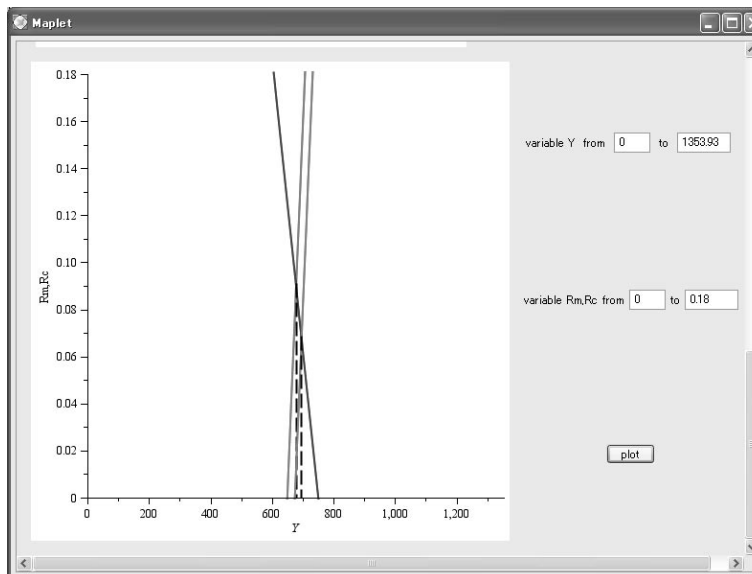


Figure 9. The two intercepts are illustrated by the two red LM curves and the blue IS curve.

For the constrained optimization problem, I use the raw Maple worksheet. For the national income determination problem, Maple functions are hidden in the teaching materials.

3. Polya's Heuristics and Deduction on Rule Databases

Polya's method, described in "How to Solve It", is a formulation of a set of heuristics. His main method includes four stages of problem solving. Polya suggested a framework for teaching meta-reasoning in mathematical problem solving. However, it is difficult to implement heuristics on computers, and in AI-research, Polya's heuristics became a challenge for automated problem solving (Melis and Ullrich 2003). Newell analyses the reasons as follows: (1) Polya's heuristics are not relevant to the real tasks of AI, (2) Human psychology, as woven into Polya's heuristics, is too different from the character of current computers, (3) Polya's methods are either already known and used or inapplicable to machine use, and so on (Newell 1983). Although Polya's methods are inapplicable to machine use, they are very effective with students. Therefore I use them in my classes.

Many researchers, including Polya, noticed that inference or reasoning ability is essential to understanding and solving problems in mathematics (Polya 1954; Polya 1969; Lakatos 1976; Betsur 2006). Betsur notes that the ultimate aim of teaching mathematics should be to help children develop mental powers so that they gain confidence in their ability to reason and justify their thinking (Betsur 2006). However, as Betsur points out, present-day education in mathematics seems to neglect teaching methods of reasoning (Betsur 2006). The important thing is to teach how to reason, although this is generally difficult. Some smart students may start from a concrete problem-solving example, generalize it, and use it as their method of reasoning systematically. However many students cannot afford to think of a reasoning method, being fully occupied with solving problems.

The inference methods in my business mathematics are deductive, not inductive. Problems in the mathematical world primarily involve deductive inference. After about 10 years of teaching, I found that the method used in a deductive database system could also be applied with students. As a database researcher, I have developed e-Learning systems for economics mathematics. I developed the instruction method based on my system construction experience and call it the "inference engine simulating method". In my classes today, I recommend that students use this method in business mathematics.

Business mathematics requires two kinds of rule databases: (1) a database of mathematics rules, and (2) a database of economics rules. The mathematics rule database includes formulae, rules, and solution plans. The economics rule database includes economic theories and concepts. An example of the latter database is illustrated in Figure 10. It is important to get students to separate common sense in economics from mathematical formulae. The two rule databases are effective in helping students pay attention to knowledge separation.

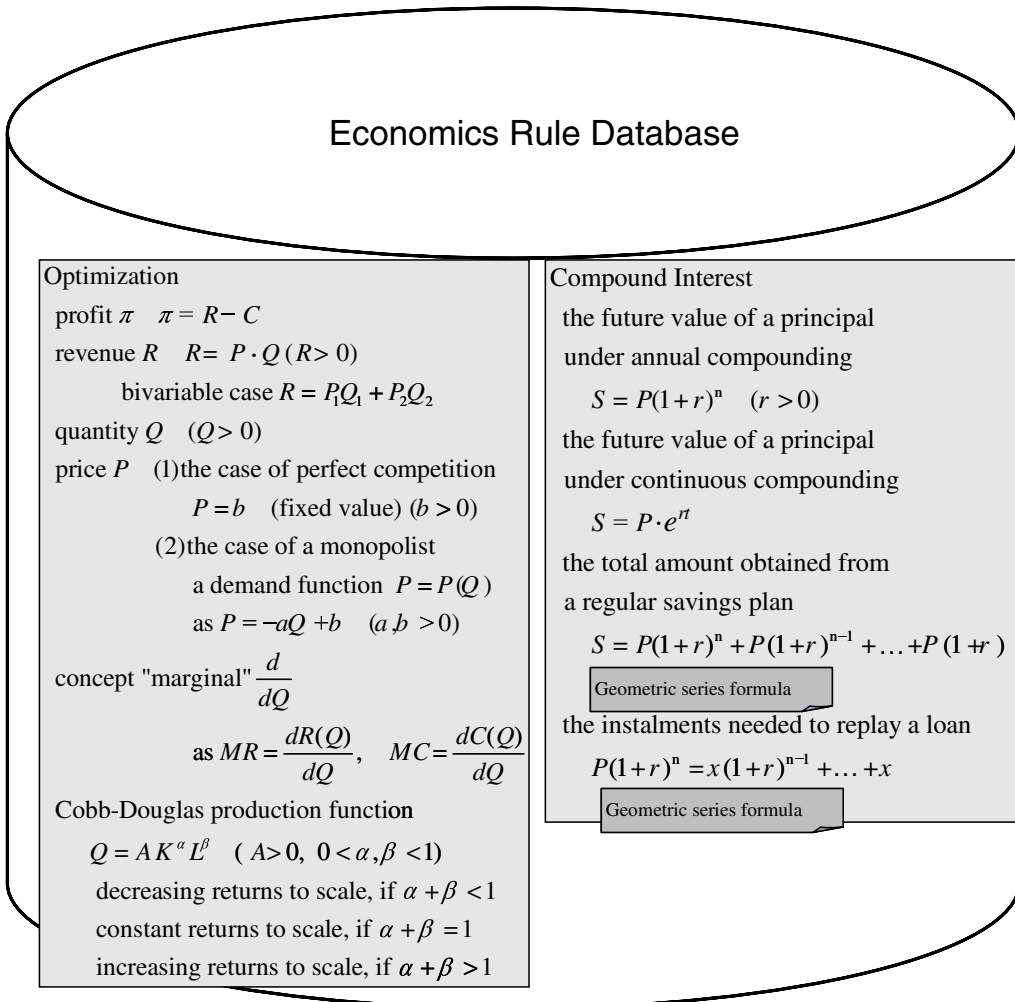


Figure 10. An Example of an economics rule database.

Now let me explain the process involved in the “inference engine simulating method”.

- (1) From the unknown in a word problem, identify several keywords and select the main one.
- (2) From the main keyword, determine strategies to find the value of the unknown.
- (3) Using the keywords, retrieve the given data list and the rule databases to collect useful rules.
- (4) Store the resultant rules in the working memory.
- (5) From the resultant rules, practice deduction.
- (6) If another keyword is found or left, conduct the search repeatedly until no further retrieval is needed.

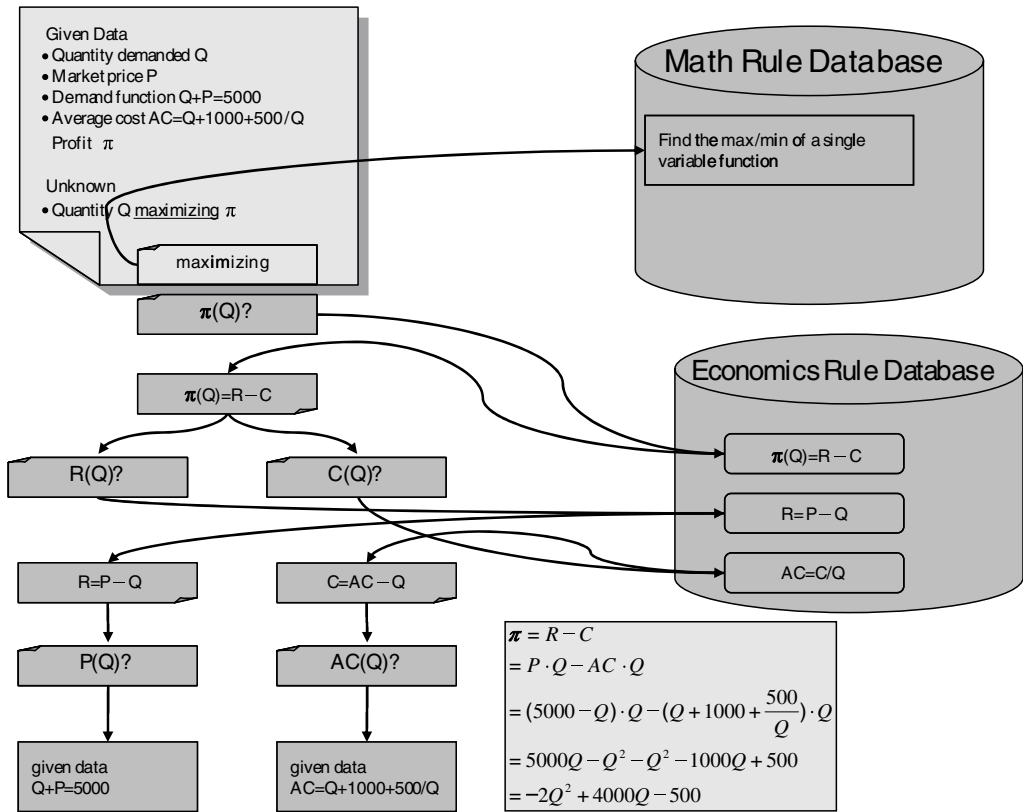


Figure 11: A sample deductive process of a profit maximizing problem.

The termination of the database search is, for example, to obtain the relationship between the unknown and the given data, or to obtain a set of simultaneous equations to be solved.

In Figure 11, an example of the deduction process is illustrated. The problem is an optimization problem, “to find the value of the quantity that maximizes the profit”, with the given data shown in the data list. In this case, the main keyword is “maximizing”. The first step is to find the mathematics rule to find the maximum/minimum of a single variable function. The next step is to try to find the expression of profit by quantity, $\pi(Q)$. The advantage of human inference is that a human can complete necessary but hidden data. In human deduction, detailed formal description is not needed because formulae can be transposed and variables substituted, and because an intelligent search then may be conducted even if many conditions are hidden. For example, a human would understand that this is a case of a single variable function of profit, not a multivariable function. In Figure 11, a rectangular tag with a page mark in the upper left corner and at the lower right, respectively, represent a keyword and a rule.

Finally, I will comment on the rote memorization of mathematical formulae. In the case of machine learning, the derived rule is also stored as a new rule so that the inference engine can skip the process to obtain the new rule, namely by using the short cut on the searching tree. This strategy is efficient in ma-

chine learning because the machine can correctly store many rules. However, unlike a computer, a student cannot memorize many rules correctly. If a student does not correctly memorize a formula and its adaptation conditions, the deduction result will be wrong. Therefore, students must memorize rules correctly.

In addition, we must be careful to pay sufficient attention to the deduction process itself. For fruitful deduction, we must annotate many keywords to one rule so that we can utilize the rule in different situations. I feel that rote memorization tends to neglect the annotation of keywords. For future problem solving, the annotation should be conducted more carefully.

4. Conclusion

In this paper, I introduced three methodologies that I practice in my business mathematics lectures. They are (1) a visual approach using Maple, (2) Polya's method described in "How to solve it", and (3) the application of deduction to rule databases. To solve a problem, a combination of both an algebraic and visual approach helps a student understand the solution process. As examples, I showed visual solution approaches to a constrained optimization problem and the curve shift effect in the national income determination problem. It is recommended that the visual solution approach be adopted in more lectures.

Polya's heuristics are effective in grasping and solving problems. It is essential for students to understand word problems before trying to solve them. So, in my lectures, for every problem, I ask, "What is the given data?" and "What is the unknown?" and make students write down the given data list and the unknown.

The third methodology is one for practising reasoning/inference in problem solving. Many researchers realize the importance of reasoning strategies in mathematics. However, it is difficult to teach reasoning strategies. The problem is a lack of efficient teaching methods. In this paper, I proposed my "inference engine simulating method" that gets students to pay sufficient attention to their deduction process. First, they define two databases - mathematics rules and economics rules ? in their heads. Next, they find the keyword from the given word problem and search the rule databases by the keyword. Students are recommended to annotate many keywords to the rule. In this paper, I showed an example of the deduction process in business mathematics. This keyword annotation method seems rather case specific because the deduction appearing there is not complicated. Therefore, application of this proposed method to generic cases will require future study.

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