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Local Powers of the MLE-Based Test for the Panel Fractional Ornstein-Uhlenbeck Process

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Abstract

We consider the testing problem associated with the panel or longitudinal fractional Ornstein-Uhlenbeck (fO-U) processes driven by independent fractional Brownian motions (fBms), where the sign of the drift parameter of each fO-U process is tested, assuming that the Hurst parameter H is known. Since the test has a trivial consistent property against a fixed alternative, we employ the local alternative hypothesis that the drift parameter is close to the null in the order of $1/(TN^\eta)$, where T is the sampling span and N is the cross section dimension with $0 < \eta < 1$. Then, for a given value of H, we compute the local power as N increases with any T.

1. Introduction

The present paper deals with the panel or longitudinal fO-U process defined on the time interval $[0, T]$. By ‘panel’ or ‘longitudinal’ it is meant in the present context that data contain multiple observations in the cross section direction at each time for each process. For this purpose let us consider the fO-U process at the ith cross section, whose stochastic differential is given by

$$dY_i(t) = \alpha_i Y_i(t)\,dt + dB_i(t), \quad Y_i(0) = 0, \quad (i = 1, \ldots, N,\ 0 \le t \le T), \tag{1}$$

where α_i ($\in R$) is an unknown drift parameter and N is the cross-section dimension, whereas $\{B_i(t)\}$ is the fBm of the known Hurst parameter $H \in (0, 1)$. It is assumed that $\{B_i(t)\}$ $(i = 1, \ldots, N)$ are independent of each other. Note that $\{B_i(t)\}$ reduces to the standard Brownian motion when $H = 1/2$. The fO-U process $\{Y_i(t)\}$ is referred to as the ergodic case when $\alpha_i < 0$, as the non-ergodic case when $\alpha_i > 0$, and as the boundary case when $\alpha_i = 0$. Note that the stochastic differential in (1) is equivalent to

$$Y_i(t) = e^{\alpha_i t} \int_0^t e^{-\alpha_i s}\,dB_i(s), \quad (i = 1, \ldots, N,\ 0 \le t \le T). \tag{2}$$

The main purpose of the present paper is to discuss the testing problem given by

$$H_0\ :\ \alpha_i = 0 \quad \text{versus} \quad H_1^L\ :\ \alpha_i < 0, \quad (i = 1, \ldots, N), \tag{3}$$

which tests against the ergodic case, or

$$H_0\ :\ \alpha_i = 0 \quad \text{versus} \quad H_1^R\ :\ \alpha_i > 0, \quad (i = 1, \ldots, N), \tag{4}$$

which tests against the non-ergodic case. Note that $Y_i(t)$ reduces to $B_i(t)$ under H_0. Of course some other alternatives are possible when $N > 1$, but we restrict our attention to the above cases because of simplicity.

The test statistic we consider here is based on the maximum likelihood estimator (MLE) of $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)'$. The likelihood for $\boldsymbol{\alpha}$ is given, from Kleptsyna and Le Breton (2002), by

$$\ell(\boldsymbol{\alpha}) = \exp\left[\sum_{i=1}^{N} \alpha_i \int_0^T Q_i(t)\, dZ_i(t) - \frac{1}{2}\sum_{i=1}^{N} \alpha_i^2 \int_0^T Q_i^2(t)\, dw(t)\right], \tag{5}$$

where

$$Q_i(t) = \frac{d}{dw(t)} \int_0^t k(t,s) Y_i(s)\, ds, \quad Z_i(t) = \int_0^t k(t,s)\, dY_i(s), \tag{6}$$

$$w(t) = \lambda^{-1} t^{2-2H}, \quad k(t,s) = \kappa^{-1}\big(s(t-s)\big)^{1/2-H}, \tag{7}$$

with $\lambda = 2H\Gamma(3-2H)\Gamma(H+1/2)/\Gamma(3/2-H)$ and $\kappa = 2H\Gamma(3/2-H)\Gamma(H+1/2)$.

The sample paths of the process $Q_i(t)$ in (6) belong to $L^2([0,T], dw)$. The process $Z_i(t)$ in (6) is a Gaussian semimartingale so that $\int_0^T Q_i(t)\, dZ_i(t)$ is the Ito integral and it has the decomposition

$$Z_i(t) = \alpha_i \int_0^t Q_i(s)\, dw(s) + M_i(t)\,, \quad M_i(t) = \int_0^t k(t,s)\, dB_i(s), \tag{8}$$

where $M_i(t)$ is the fundamental martingale with the quadratic variation $w(t)$ in (7) and $\mathrm{Cov}(M_i(s), M_i(t)) = \lambda^{-1}\big(\min(s,t)\big)^{2-2H}$, which was discussed in Norros et al. (1999) in connection with approximating $B_i(t)$ by $M_i(t)$.

When $N = 1$, this testing problem was earlier discussed in Moers (2012), Tanaka (2013, 2015), and Kukush et al. (2017). One important necessity for the present test is that the methods of constructing the estimators and their asymptotic properties essentially depend on the sign of the drift parameter (Kukush et al. (2017)). In connection with econometric problems, the present test is referred to as the unit root test. This is because of the relationship between the fO-U process and the discrete-time near-unit root process defined by

$$y_j = \rho y_{j-1} + v_j, \quad \rho = 1 + \frac{\alpha}{n}, \quad y_0 = 0, \quad (j = 1, ..., n), \tag{9}$$

where the error process $\{v_j\}$ is a stationary long-memory process generated by

$$v_j = (1-L)^{-(H-1/2)}\varepsilon_j = \sum_{k=0}^{\infty} \frac{\Gamma(k+H-1/2)}{\Gamma(H-1/2)\Gamma(k+1)}\, \varepsilon_{j-k}, \tag{10}$$

with $1/2 < H < 1$, L being the lag-operator, and $\{\varepsilon_j\} \sim \text{i.i.d.}(0, \sigma^2)$. Then it holds (Davydov (1970)) that

$$\frac{\text{const.}}{n^H}\, y_{[nt]} \;\Rightarrow\; e^{\alpha t} \int_0^t e^{-\alpha s}\, dB(s), \quad (0 \le t \le 1),$$

where $\Rightarrow$ signifies weak convergence as $n \to \infty$. The discrete-time process $\{y_j\}$ in (9) is said to have a unit root when $\alpha = 0$. Thus the testing problem for $\alpha = 0$ is called the unit root test. Note that the parameter α in (9) plays the same role as the drift parameter α_i in the fO-U process in (1) or (2).

Returning to the panel fO-U process in (1), suppose first that $N = 1$ and let us proceed without the subsript i for each quantity. Then the MLE of α is given for any $H \in (0, 1)$ by

$$\tilde{\alpha}(T) = \frac{\int_0^T Q(t)\, dZ(t)}{\int_0^T Q^2(t)\, dw(t)} = \alpha + \frac{\int_0^T Q(t)\, dM(t)}{\int_0^T Q^2(t)\, dw(t)}. \tag{11}$$

In particular, when $\alpha = 0$, we have $Y(t) = B(t)$ and the self-similarity property of $B(t)$ yields

$$T\tilde{\alpha}(T) = \frac{\int_0^1 Q(t)\, dM(t)}{\int_0^1 Q^2(t)\, dw(t)}, \qquad (\alpha = 0), \tag{12}$$

which shows that the distribution of $T\tilde{\alpha}(T)$ does not depend on T when $\alpha = 0$.

Although $\tilde{\alpha}(T)$ is defined in the same way for any α, unlike the least squares estimator (LSE), its asymptotic distribution as $T \to \infty$ is different among the signs of α because it holds that

$$\tilde{\alpha}(T) - \alpha = \begin{cases} O_p(T^{-1/2}) & (\alpha < 0), \\ O_p(e^{-\alpha T}) & (\alpha > 0), \\ O_p(T^{-1}) & (\alpha = 0), \end{cases} \qquad T\,\tilde{\alpha}(T) \;\to\; \begin{cases} -\infty & (\alpha < 0), \\ +\infty & (\alpha > 0), \\ O_p(1) & (\alpha = 0). \end{cases} \tag{13}$$

More specifically, it holds that, for $\alpha < 0$,

$$\sqrt{T}(\tilde{\alpha}(T) - \alpha) \;\Rightarrow\; \mathrm{N}(0,\, -2\alpha),$$

which was proved in Kleptsyna and Le Breton (2002), whereas, for $\alpha > 0$,

$$\frac{e^{\alpha T}(\tilde{\alpha}(T) - \alpha)}{2\alpha} \;\Rightarrow\; \sqrt{\sin \pi H} \times C(0, 1),$$

where $C(0, 1)$ is a standard Cauchy random variable, which was proved in Tanaka (2015). Then we have

$$T\tilde{\alpha}(T) = \begin{cases} \sqrt{T}\sqrt{T}(\tilde{\alpha}(T) - \alpha) + T\alpha = -O(T) & (\alpha < 0), \\ 2\alpha T e^{-\alpha T} \dfrac{e^{\alpha T}(\tilde{\alpha}(T) - \alpha)}{2\alpha} + T\alpha = O(T) & (\alpha > 0), \\ O_p(1) & (\alpha = 0). \end{cases}$$

It follows that the test based on $\tilde{\alpha}(T)$ is consistent and has trivial limting powers as $T \to \infty$ against $\alpha < 0$ and $\alpha > 0$. Moreover we have the following theorem.

Theorem 1. For the fO-U process in (1) with $N = 1$, let us consider the testing problems (3) and (4). Then the powers of the tests based on $\tilde{\alpha}(T)$ in (11) do not depend on each value of α and T, but on $\alpha \times T$.

It follows from Theorem 1 that the present test is consistent and has trivial limiting powers as $T \to \infty$ or $|\alpha| \to \infty$. Because of this fact there is no point in examining limiting powers as $T \to \infty$ or $|\alpha| \to \infty$. Instead we pursue limiting local powers under a sequence of local alternatives which depend on the sampling interval T and the cross section dimension N.

In Section 2, dealing with the panel fO-U process in (1), we consider the local alternative $H_1 : \alpha_i = \delta_N/T$, where $\delta_N = \delta/N^\eta$ with δ being a fixed constant and $\eta \in (0,1)$. We present a feasible way of computing finite sample local powers as well as limiting local powers of the MLE-based test as $N \to \infty$. Section 3 demonstrates that the MLE-based test is asymptotically efficient. Section 4 presents graphically the local powers at the 5% level for any value of T and various values of N including $N = \infty$. Section 5 concludes this paper, where the difficulty of the LSE-based test is mentioned together with an extension of the present test. The proofs of theorems are given in the Appendix.

2. Powers under the local alternative

In this section we deal with the testing problems in (3) and (4), where we consider the local alternatives given by

$$H_1 \,:\, \alpha_i = \frac{\delta_N}{T}, \quad \delta_N = \frac{\delta}{N^\eta}, \tag{14}$$

where δ is a constant with $\delta < 0$ being the ergodic alternative and $\delta > 0$ being the non-ergodic alternative, whereas $\eta \in (0,1)$ is determined later so that the limiting power becomes nontrivial as $N \to \infty$.

We consider a test based on the MLE of $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)'$. For this purpose the MLE of $\boldsymbol{\alpha}$ is computed under the assumption that $\alpha_i = \alpha$ for $i = 1, \ldots, N$. Then the MLE of α is given from (5) as

$$\tilde{\alpha}(N,T) = \frac{\sum_{i=1}^N \int_0^T Q_i(t)\, dZ_i(t)}{\sum_{i=1}^N \int_0^T Q_i^2(t)\, dw(t)} = \frac{\sum_{i=1}^N U_i(T)}{\sum_{i=1}^N V_i(T)}, \tag{15}$$

where

$$U_i(T) = \int_0^T Q_i(t)\, dZ_i(t), \qquad V_i(T) = \int_0^T Q_i^2(t)\, dw(t). \tag{16}$$

To compute the distribution of $\tilde{\alpha}(N,T)$ the following property is useful.

Theorem 2. It holds that, under $H_0 : \alpha = 0$ and $H_1 : \alpha_i = \alpha = \delta_N/T$,

$$T\tilde{\alpha}(N,T) = \frac{\dfrac{1}{T}\displaystyle\sum_{i=1}^N \int_0^T Q_i(t)\, dZ_i(t)}{\dfrac{1}{T^2}\displaystyle\sum_{i=1}^N \int_0^T Q_i^2(t)\, dw(t)} \overset{\mathcal{D}}{=} \frac{\displaystyle\sum_{i=1}^N \int_0^1 Q_i(t)\, dZ_i(t)}{\displaystyle\sum_{i=1}^N \int_0^1 Q_i^2(t)\, dw(t)} = \frac{\sum_{i=1}^N U_i(1)}{\sum_{i=1}^N V_i(1)}, \tag{17}$$

where '$\overset{\mathcal{D}}{=}$' stands for the distributional equivalence.

We now use $T\tilde{\alpha}(N,T)$ as a test statistic. For the ergodic alternative with $\alpha < 0$, the null hypothesis of $\alpha = 0$ is rejected when $T\tilde{\alpha}(N,T)$ is small and the power of the test at the 100 γ% is computed as $P(T\tilde{\alpha}(N,T) < z_\gamma)$, where z_γ is the 100γ% point of the null distribution of $T\tilde{\alpha}(N,T)$. The following theorem describes how to compute the local power against the ergodic alternative when N is finite.

Theorem 3. The power of the present test at the $100\gamma\%$ level against $H_1^L : \alpha = \delta_N/T$ with $\delta_N < 0$ is computed as

$$P(T\tilde{\alpha}(N,T) < z_\gamma) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{\theta}\mathrm{Im}[\{m(-i\theta, i\theta z_\gamma)\}^N]\, d\theta. \tag{18}$$

Here z_γ is the $100\gamma\%$ point of the null distribution of $T\tilde{\alpha}(N,T)$ and $m(\theta_1,\theta_2)$ is the joint moment generating function (m.g.f.) of $U_i(1)$ and $V_i(1)$ given by

$$\begin{aligned} m(\theta_1,\theta_2) &= \mathrm{E}\Big[\exp\{\theta_1 U_i(1) + \theta_2 V_i(1)\}\Big] \\ &= e^{-(\delta_N+\theta_1)/2}\Bigg[\left(1+\frac{(\delta_N+\theta_1)^2}{\mu^2}\right)\cosh^2\frac{\mu}{2} - \frac{\delta_N+\theta_1}{\mu}\sinh\mu \\ &\qquad + \frac{\pi}{4\sin\pi H}\Bigg\{-\frac{(\delta_N+\theta_1)^2}{\mu}I_{-H}\left(\frac{\mu}{2}\right)I_{H-1}\left(\frac{\mu}{2}\right) \\ &\qquad + \mu I_{-H+1}\left(\frac{\mu}{2}\right)I_H\left(\frac{\mu}{2}\right)\Bigg\}\Bigg]^{-1/2}, \end{aligned} \tag{19}$$

where $\mu = \sqrt{\delta_N^2 - 2\theta_2}$, whereas $I_\nu(z)$ is the modified Bessel function of the first kind defined by

$$I_\nu(z) = \sum_{k=0}^\infty \frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}.$$

The local power against the non-ergodic alternative at the $100\gamma\%$ level is computed as $P(T\tilde{\alpha}(N,T) > z_{1-\gamma})$. The m.g.f. in (19) was first obtained in Kleptsyna and Le Breton (2002) (see also Tanaka (2015)). It is noticed in (19) that the m.g.f. remains the same when H is replaced by $1-H$. This means that the MLE under H is the same as that under $1-H$. Thus the MLE can be applied to $0 < H < 1$, and the distribution of the MLE is symmetric around $H = 1/2$.

The powers of the test at the 5% level will be presented in Section 4 for various values of N including $N = \infty$, paying attention to the effect of the Hurst index H.

We next discuss how to compute the limiting power of the test as $N \to \infty$. For this purpose we have

Theorem 4. Consider $T\tilde{\alpha}(N,T) = \sum_{i=1}^N U_i(1)/\sum_{i=1}^N V_i(1)$ under $\alpha = \delta_N/T$. When $\delta_N \to 0$ as $N \to \infty$ with any T, it holds that

$$\mathrm{E}(U_i(1)) = \left.\frac{\partial m(\theta_1,0)}{\partial\theta_1}\right|_{\theta_1=0} = \beta_H\delta_N + O(\delta_N^2), \tag{20}$$

$$\mathrm{E}(U_i^2(1)) = \left.\frac{\partial^2 m(\theta_1,0)}{\partial\theta_1^2}\right|_{\theta_1=0} = \beta_H + 3\gamma_H\delta_N + O(\delta_N^2), \tag{21}$$

$$\mathrm{E}(V_i(1)) = \left.\frac{\partial m(0,\theta_2)}{\partial\theta_2}\right|_{\theta_2=0} = \beta_H + \gamma_H\delta_N + O(\delta_N^2), \tag{22}$$

where

$$\beta_H = \frac{1}{4} + \frac{1}{16H(1-H)}, \quad \gamma_H = \frac{1}{12} + \frac{1}{16H(1-H)}.$$

We note that the above results for $H = 1/2$ were earlier obtained for discrete-time panel unit root models in econometrics (Tanaka (2017)). Let us consider the situation where $N \to \infty$. It follows from Theorem 4, the law of large numbers (LLN) and the central limit theorem (CLT) that, as $N \to \infty$,

$$\frac{1}{N}\sum_{i=1}^{N} V_i(1) \ \to \ \beta_H \text{ in probability}, \tag{23}$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} U_i(1) \ \Rightarrow \ \mathrm{N}(\sqrt{N}\delta_N\beta_H,\, \beta_H), \tag{24}$$

which leads us to establish

Theorem 5. Suppose that $\delta_N = \delta/\sqrt{N} = T\alpha$, where δ is a constant. Then it holds that, as $N \to \infty$ with any T, it holds that

$$\sqrt{N}T\tilde{\alpha}(N,T) = \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N} U_i(1)}{\frac{1}{N}\sum_{i=1}^{N} V_i(1)} \ \Rightarrow \ \mathrm{N}\left(\delta,\, \frac{1}{\beta_H}\right), \tag{25}$$

$$P(\sqrt{N}T\sqrt{\beta_H}\tilde{\alpha}(N,T) \le z_\gamma) \ \to \ \Phi(z_\gamma - \sqrt{\beta_H}\delta), \tag{26}$$

where $\Phi(\cdot)$ is the distribution function of N(0,1).

It follows from Theorem 5 that the local alternative which yields a nontrivial result is of the form $\alpha = \delta/(T\sqrt{N})$ and the limiting local power as $N \to \infty$ can be computed from (26). Note that the asymptotic normality holds under the local alternative with any negative or positive δ. This contrasts with the case of the fixed alternative with a positive constant α, for which the statistic tends to Cauchy distribution.

The next section demonstrates that the MLE-based test discussed in this section is asymptotically efficient.

3. Asymptotic efficiency of the MLE-based test

Suppose that the panel fractional O-U model is given by

$$dY_i(t) = \alpha_i Y_i(t)\,dt + dB_i(t), \quad \alpha_i = \frac{\delta}{T\sqrt{N}}, \tag{27}$$

and consider the testing problem

$$H_0 \ : \ \alpha_i = 0 \quad \text{vs.} \quad H_1 \ : \ \alpha_i = \frac{\theta}{T\sqrt{N}}, \quad (i = 1,\, \ldots,\, N), \tag{28}$$

where θ is a given constant. This is a test against a simple alternative.

Then the Neyman-Pearson lemma tells us that the test rejects H_0 when

$$\ell\left(\frac{\theta}{T\sqrt{N}}\right)-\ell(0)$$

takes large values is the most powerful (MP), where $\ell(\alpha)$ is the likelihood for α given by

$$\ell(\alpha)=\exp\left[\alpha\sum_{i=1}^{N}\int_0^T Q_i(t)\,dZ_i(t)-\frac{\alpha^2}{2}\sum_{i=1}^{N}\int_0^T Q_i^2(t)\,dw(t)\right].$$

Thus the MP test rejects H_0 when

$$\begin{aligned}S_{NT}(\theta) &= \frac{\theta}{T\sqrt{N}}\sum_{i=1}^{N}\int_0^T Q_i(t)\,dZ_i(t)-\frac{\theta^2}{2T^2N}\sum_{i=1}^{N}\int_0^T Q_i^2(t)\,dw(t)\\ &\overset{\mathcal{D}}{=} \frac{\theta}{\sqrt{N}}\sum_{i=1}^{N}\int_0^1 Q_i(t)\,dZ_i(t)-\frac{\theta^2}{2N}\sum_{i=1}^{N}\int_0^1 Q_i^2(t)\,dw(t)\end{aligned}$$

takes large values. It follows from (23) and (24) that the MP statistic $S_{NT}(\theta)$ converges to

$$S_{NT}(\theta)\ \Rightarrow\ \theta\sqrt{\beta_H}\,X-\frac{\theta^2\beta_H}{2},\quad X\sim \mathrm{N}\left(\delta\sqrt{\beta_H},\,1\right). \tag{29}$$

Thus it holds that

$$\begin{aligned}P\left(\frac{S_{NT}(\theta)+\theta^2\beta_H/2}{\theta\sqrt{\beta_H}}<z_\gamma\right) &\to P\left(X<z_\gamma\right)\\ &= \Phi\left(z_\gamma-\delta\sqrt{\beta_H}\right).\end{aligned} \tag{30}$$

It is seen that the power function does not depend on θ, which implies that the MP test based on $S_{NT}(\theta)$ is UMP, and the MLE-based test is asymptotically efficient because the two power functions coincide.

4. Computation of local powers

Here we report powers of our tests against the ergodic and non-ergodic alternatives at the 5% significance level. For this purpose we first compute the 5% and 95% points of the null distribution of the statistic $T\tilde{\alpha}(N,T)$ for various values of N and H. These can be obtained from (18) by putting $\delta_N=0$. Table 1 reports these percent points for $N=$ 1, 10, 50 and $H=0.5$, 0.7, 0.9. It is recognized that, for each N, the statistic becomes slightly more concentrated as H gets away from $H=0.5$. Note that the distirbutions remain unchanged with H replaced by $1-H$.

Table 1. 5% and 95% points of the distribution of $T\tilde{\alpha}(N,T)$ under $\alpha = 0$

(N, H)								
(1, 0.5)	(1, 0.7)	(1, 0.9)	(10, 0.5)	(10, 0.7)	(10, 0.9)	(50, 0.5)	(50, 0.7)	(50, 0.9)
				5%				
−8.039	−7.964	−7.415	−1.100	−1.057	−0.805	−0.388	−0.371	−0.279
				95%				
1.285	1.250	1.084	0.560	0.537	0.427	0.287	0.274	0.213

Figure 1 draws the null densities of $\sqrt{N}T\tilde{\alpha}(N,T)$ for $N = 1$ with $H = 0.5, 0.7, 0.9$ and Figure 2 for $N = 50$ with the same values of H. Note that we have added $\sqrt{N}$ to the factor so that the distribution does not depend on N asymptotically. In fact it follows from Theorem 5 that $\sqrt{N}T\tilde{\alpha}(N,T) \Rightarrow \mathrm{N}(0, 1/\beta_H)$ as $N \to \infty$. It is seen from these figures that

(a) When $N = 1$, the distribution is far from normal. The distribution for $N = 1$ with $H = 0.5$ is called the unit root distribution in econometrics. As N becomes large, however, it tends to normal with the mean 0 because of the CLT.

(b) For fixed N, the distribution becomes slightly more concentrated as H gets away from $H = 0.5$, as was also recognized in Table 1.

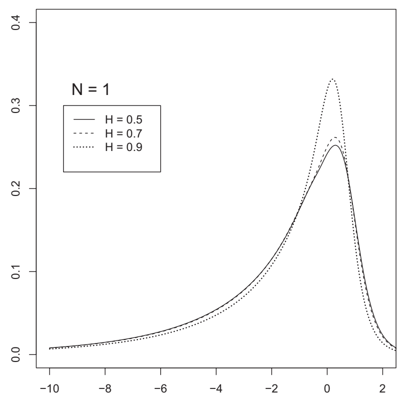

Figure 1

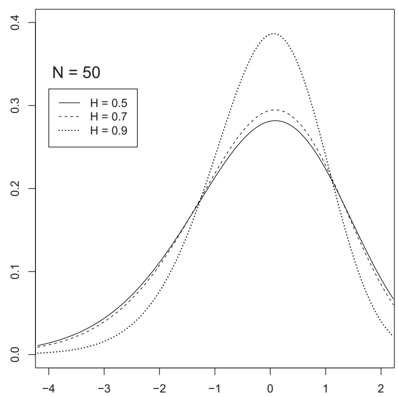

Figure 2

Figure 3 shows the densities of $\sqrt{N}T\tilde{\alpha}(N,T)$ under the local ergodic alternative $\alpha = \delta/(\sqrt{N}T)$ with $\delta = -1$ and $N = 1$, whereas Figure 4 shows those densities with $\delta = -1$ and $N = 50$. When $N = 1$, these densities are still skewed to the left, but not so much as the null densities shown in Figure 1. When $N = 50$, these are quite close to the density of N$(-1,\, 1/\beta_H)$, where $\delta = -1$ and $\beta_H = 1/4 + 1/(16H(1-H))$. It follows that the distribution becomes more concentrated as H gets away from $H = 0.5$. Figure 5 draws the corresponding densities under the local non-ergodic alternative with $\delta = 1$ and $N = 50$. These densities are close to those of N$(1,\, 1/\beta_H)$.

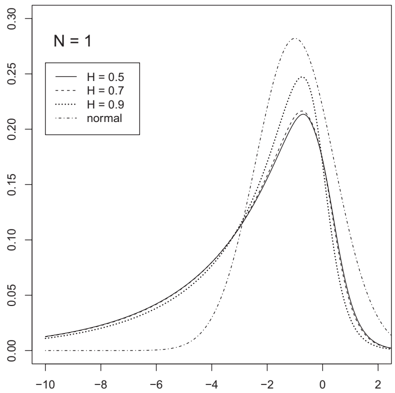

Figure 3

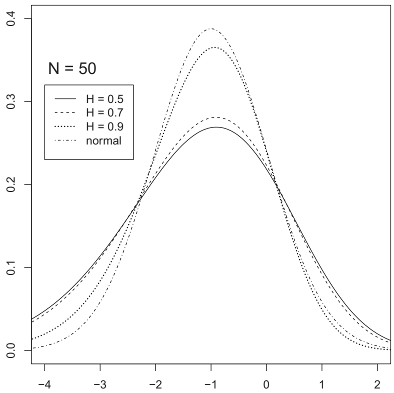

Figure 4

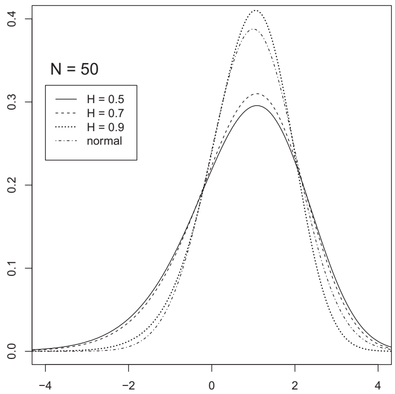

Figure 5

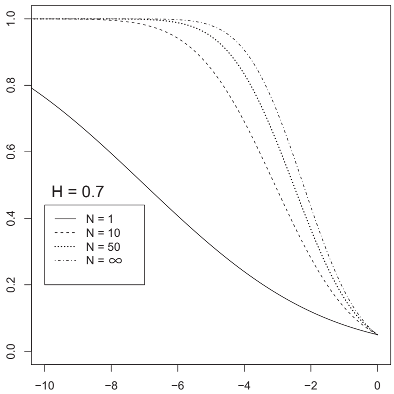

Figure 6

Figure 6 presents the local powers of the test against the ergodic alternative with $H = 0.7$. The powers for $N < \infty$ were computed from (26). The limiting power for $N = \infty$ is also shown, which was computed from (18). It is seen that the finite sample powers increase with N and converge to the limiting power from below. This is not trivial because the local alternative with $\delta_N = \delta/\sqrt{N}$ is in the neighborhood of the null in the order of $1/\sqrt{N}$, which converges to the null. The fact that the covergence of finite sample powers to the limiting power from below is specific to the ergodic alternative. It will be seen in Figure 8 that the finite sample powers against the non-ergodic alternative converge from above to the limiting power. Figure 7 compares, among H, the finite sample performance of the test against the ergodic alternative when $N = 10$. It is seen that the test is more powerful when H gets away from $H = 0.5$.

Figure 8 shows the finite sample and limiting local powers against the non-ergodic alternative with $H = 0.7$. It is seen that the power performance is different between the ergodic and non-ergodic alternatives. The test is more powerful against the non-ergodic. It is also seen that the finite sample powers converge to the limiting power from above, as was mentioned before.

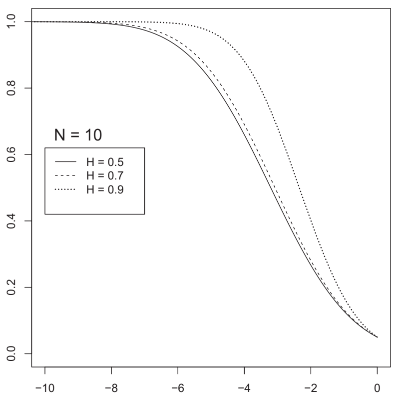

Figure 7

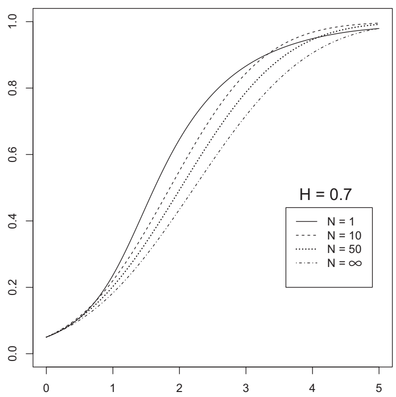

Figure 8

5. Concluding remarks

We have considered the MLE-based test for the sign of the drift parameter in the panel fO-U process with the Hurst index H known, which tests for the process to be the fBm against the ergodic or non-ergodic fO-U process. Because the test is consistent against the fixed alternative as the cross section dimension $N \to \infty$, we have assumed a local alternative close to the null in the order of $T^{-1}N^{-\eta}$, where T is any time span and $0 < \eta < 1$. It was found that $\eta = 1/2$ yields a non-trivial power as $N \to \infty$. It was also demonstrated that the MLE-based test is asymptotically efficient in the sense that the power function of the MP test coincides with that of the MLE-based test. The power performance of the test was examined for various values of N including $N = \infty$ by paying attention to the effect of the value of H.

The present model may be extended to the fractional Vasicek model

$$dY_i(t) = (\alpha_i Y_i(t) + \mu_i)\,dt + dB_i(t), \quad Y_i(0) = 0, \quad (i = 1, \ldots, N,\ 0 \le t \le T).$$

For the time series case with $N = 1$, the MLEs of $\mu = \mu_i$ and $\alpha = \alpha_i$ were discussed in Tanaka et al. (2020).

It is also interesting to compare the power of the present test with that of the LSE-based test. Tanaka (2020) compared the LSE and MLE in the case of $N = 1$. Suppose that $H \in (1/2, 1)$. Then, assuming $\alpha = \alpha_i$ $(i = 1, \ldots, N)$, the LSE of α in (1) is given by

$$\hat{\alpha} = \frac{\sum_{i=1}^N \int_0^T Y_i(t)\, dY_i(t)}{\sum_{i=1}^N \int_0^T Y_i^2(t)\, dt} = \frac{\sum_{i=1}^N Y_i^2(T)/2}{\sum_{i=1}^N \int_0^T Y_i^2(t)\, dt},$$

where the integral in the numerator is of Riemann-Stieltjes type. It might be argued that the power of the LSE-based test could be computed in the same way as that of the MLE-based test. This, however, is not the case because the joint m.g.f. of $Y_i^2(T)$ and $\int_0^T Y_i^2(t)\, dt$ has never been derived even under the null $H_0 : \alpha = 0$. In fact, the joint m.g.f. of $B_i^2(T)$ and $\int_0^T B_i^2(t)\, dt$ is unknown, although an approximation was suggested in Tanaka (2014). This is a topic for future research.

In the present paper the cross-sectional independence was assumed, that is, the fractional Brownian motions $B_1(t), \ldots, B_N(t)$ which generate the panel fO-U processes are independent of each other. An extension to the cross-sectional dependence is also another topic to be pursued.

Appendix

Proof of Theorem 1: Let $x_\gamma(T)$ be the $100\gamma\%$ point of the null distribution of $\tilde{\alpha}(T)$ under the sampling interval T. Then the power of the test against $\alpha < 0$ at the significance level γ is computed, by using the formula in Imhof (1961), as

$$\begin{aligned} P(\tilde{\alpha}(N,T) < x_\gamma(T)) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{\theta}\mathrm{Im}\Big[m\Big(-i\theta,\, i\theta x_\gamma(T)\Big)\Big]\, d\theta \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{u}\mathrm{Im}\Big[m\Big(-iu/T,\, iux_\gamma(T)/T\Big)\Big]\, du, \end{aligned}$$

where $m(\theta_1, \theta_2)$ is the m.g.f. of $U_i(T)$ and $V_i(T)$ in (16), which is given by Kleptsyna and Le Breton (2002) as

$$\begin{aligned} m(\theta_1, \theta_2) = e^{-T(\alpha+\theta_1)/2} \Bigg[& \left(1 + \frac{(\alpha+\theta_1)^2}{\mu^2}\right)\cosh^2\frac{\mu T}{2} - \frac{\alpha+\theta_1}{\mu}\sinh \mu T \\ & + \frac{\pi T}{4\sin \pi H}\left\{ -\frac{(\alpha+\theta_1)^2}{\mu} I_{-H}\left(\frac{\mu T}{2}\right) I_{H-1}\left(\frac{\mu T}{2}\right) \right. \\ & \left. + \mu I_{-H+1}\left(\frac{\mu T}{2}\right) I_H\left(\frac{\mu T}{2}\right)\right\}\Bigg]^{-1/2}, \end{aligned} \qquad \text{(A.1)}$$

where $\mu = \sqrt{\alpha^2 - 2\theta_2}$. Then it holds that

$$\begin{aligned} g(u) &= m\Big(-iu/T,\, iux_\gamma(T)/T\Big) \\ &= e^{(iu-\alpha T)/2}\Bigg[\left(1 + \frac{(\alpha T - iu)^2}{4\xi^2}\right)\cosh^2 \xi - \frac{\alpha T - iu}{2\xi}\sinh 2\xi \\ &\quad + \frac{\pi}{4\sin \pi H}\left\{ -\frac{(\alpha T - iu)^2}{2\xi} I_{-H}(\xi) I_{H-1}(\xi) + 2\xi I_{-H+1}(\xi) I_H(\xi)\right\}\Bigg]^{-1/2}, \end{aligned}$$

where

$$\xi = \frac{1}{2}\sqrt{(\alpha T)^2 - 2iuTx_\gamma(T)} = \frac{1}{2}\sqrt{(\alpha T)^2 - 2iux_\gamma(1)}.$$

This last equality, that is, $Tx_\gamma(T) = x_\gamma(1)$, comes from (12). Then it is seen from the form of $g(u)$ that the power depends only on $\alpha \times T$, and does not depend on each value of α and T, which establishes Theorem 1.

Proof of Theorem 2: Let us denote the m.g.f. $m(\theta_1, \theta_2)$ in (A.1) as $m(\theta_1, \theta_2;\, \alpha, T)$ to express its dependence on α and T explicitly. Then it can be checked that, for $\alpha = \delta_N/T$, the joint m.g.f. of $\sum_{i=1}^N U_i(T)/T$ and $\sum_{i=1}^N V_i(T)/T^2$ is given by

$$m_N(\theta_1, \theta_2) = \left[m(\theta_1/T, \theta_2/T^2;\, \delta_N/T, T)\right]^N = \left[m(\theta_1, \theta_2;\, \delta_N, 1)\right]^N,$$

which means that the joint distribution of $\sum_{i=1}^N U_i(T)/T$ and $\sum_{i=1}^N V_i(T)/T^2$ with $\alpha = \delta_N/T$ is the same as that of $\sum_{i=1}^N U_i(1)$ and $\sum_{i=1}^N V_i(1)$ with $\alpha = \delta_N$. This also holds for $\alpha = 0$. Thus Theorem 2 is established.

Proof of Theorem 3: It follows from Theorem 2 that

$$P(T\tilde{\alpha}(N,T) < z_\gamma) = P\left(\sum_{i=1}^N U_i(1) < z_\gamma \sum_{i=1}^N V_i(1)\right),$$

which yields (18) because of the formula in Imhof (1961). Then Theorem 3 is established from Kleptsyna and Le Breton (2002).

Proof of Theorem 4: Let $m(\theta_1, \theta_2)$ be the joint m.g.f. of $U_i(1)$ and $V_i(1)$ under $\alpha = \delta_N/T$, which is given in (19). Then we have, for $\delta_N \neq 0$,

$$\begin{aligned} m(\theta, 0) \quad = \quad & e^{-(\delta_N+\theta)/2}\Bigg[\left(1 + \frac{(\delta_N+\theta)^2}{\delta_N^2}\right)\cosh^2\frac{\delta_N}{2} - \frac{\delta_N+\theta}{\delta_N}\sinh\delta_N \\ & + \frac{\pi}{4\sin\pi H}\Bigg\{-\frac{(\delta_N+\theta)^2}{\delta_N} I_{-H}\left(\frac{\delta_N}{2}\right) I_{H-1}\left(\frac{\delta_N}{2}\right) \\ & + \delta_N I_{-H+1}\left(\frac{\delta_N}{2}\right) I_H\left(\frac{\delta_N}{2}\right)\Bigg\}\Bigg]^{-1/2}, \end{aligned}$$

$$\begin{aligned} \mathrm{E}(U_i(1)) \quad &= \quad \left.\frac{d\,m(\theta,0)}{d\,\theta}\right|_{\theta=0} \\ &= -\frac{1}{2} - \frac{1}{2}e^{\delta_N}\left(\frac{1}{\delta_N} + \frac{1}{\delta_N}e^{-\delta_N} - \frac{\pi}{2\sin\pi H} I_{-H}\left(\frac{\delta_N}{2}\right) I_{H-1}\left(\frac{\delta_N}{2}\right)\right), \\ &= -\frac{1}{2}\left[1 + \frac{1}{\delta_N} + e^{\delta_N}\left(\frac{1}{\delta_N} - \frac{\pi}{2\sin\pi H} I_{-H}\left(\frac{\delta_N}{2}\right) I_{H-1}\left(\frac{\delta_N}{2}\right)\right)\right] \\ &= \beta_H\delta_N + O(\delta_N^2), \end{aligned}$$

where we have used the relations

$$\begin{aligned} \frac{\pi}{\sin\pi H} \quad &= \quad \Gamma(H)\Gamma(1-H), \\ I_{-H}\left(\frac{\delta_N}{2}\right) I_{H-1}\left(\frac{\delta_N}{2}\right) \quad &= \quad \frac{1}{\Gamma(H)\Gamma(1-H)}\left(\left(\frac{\delta_N}{4}\right)^{-1} + \frac{\delta_N}{4H(1-H)}\right) + O(\delta_N^3). \end{aligned}$$

Simlilarly, we have

$$\begin{aligned}
\mathrm{E}(U_i^2(1)) &= \left.\frac{d^2 m(\theta,0)}{d\,\theta^2}\right|_{\theta=0} \\
&= -\frac{1}{2}A + \frac{1}{4}e^{\delta_N}B - \frac{1}{2}e^{\delta_N}C + \frac{3}{4}e^{2\delta_N}B^2,
\end{aligned}$$

where

$$\begin{aligned}
A &= \mathrm{E}(U_i(1)) = \beta_N\delta_N + O(\delta_N^2), \\
B &= \frac{1}{\delta_N} + e^{-\delta_N} - \frac{\pi}{2\sin\pi H} I_{-H}\left(\frac{\delta_N}{2}\right) I_{H-1}\left(\frac{\delta_N}{2}\right), \\
C &= \frac{1}{\delta_N^2}(1+\cosh\delta_N) - \frac{1}{\delta_N}\frac{\pi}{2\sin\pi H} I_{-H}\left(\frac{\delta_N}{2}\right) I_{H-1}\left(\frac{\delta_N}{2}\right),
\end{aligned}$$

which yields the expression for $\mathrm{E}(U_i^2(1))$ in (21). To prove (22), we consider

$$\begin{aligned}
m(0,\theta) &= e^{-\delta_N/2}\left[\left(1+\frac{\delta_N^2}{\mu^2}\right)\cosh^2\frac{\mu}{2} - \frac{\delta_N}{\mu}\sinh\mu \right. \\
&\qquad + \frac{\pi}{4\sin\pi H}\left\{-\frac{\delta_N^2}{\mu} I_{-H}\left(\frac{\mu}{2}\right) I_{H-1}\left(\frac{\mu}{2}\right)\right. \\
&\qquad \left.\left. + \mu I_{-H+1}\left(\frac{\mu}{2}\right) I_H\left(\frac{\mu}{2}\right)\right\}\right]^{-1/2},
\end{aligned}$$

where $\mu = \sqrt{\delta_N^2 - 2\theta}$. Then we obtain, for $\delta_N > 0$,

$$\begin{aligned}
\mathrm{E}(V_i(1)) &= \left.\frac{d\,m(0,\theta)}{d\,\theta}\right|_{\theta=0} \\
&= -\frac{1}{2}e^{\delta_N}\Big[\frac{1}{\delta_N^2} + \left(\frac{1}{\delta_N} + \frac{1}{\delta_N^2}\right)e^{-\delta_N} + \frac{\pi}{4\sin\pi H}\Big\{-\frac{1}{\delta_N}(I_{-H}I_{H-1} + I_{-H+1}I_H) \\
&\qquad + \frac{1}{4}\Big((I_{-H-1} + I_{-H+1})I_{H-1} + (I_{H-2} + I_H)I_{-H} \\
&\qquad - (I_{-H} + I_{-H+2})I_H - (I_{H-1} + I_{H+1})I_{-H+1}\Big)\Big\}\Big],
\end{aligned}$$

where $I_\nu = I_\nu(\delta_N/2)$ and we have used the relations.

$$\frac{d\,I_\nu(z)}{d\,z} = \frac{1}{2}\Big(I_{\nu-1}(z) + I_{\nu+1}(z)\Big), \quad \left.\frac{d\,\mu}{d\,\theta}\right|_{\theta=0} = -\frac{1}{\delta_N}.$$

Noting further that

$$\begin{aligned}
I_{-H}I_{H-1} &= \frac{1}{\Gamma(H)\Gamma(1-H)}\Big(\frac{4}{\delta_N} + \frac{\delta_N}{4H(1-H)}\Big) + O(\delta_N^3), \\
I_{-H+1}I_H &= \frac{1}{\Gamma(H)\Gamma(1-H)}\frac{\delta_N}{4H(1-H)} + O(\delta_N^3), \\
I_{-H-1}I_{H-1} &= -\frac{4H}{\Gamma(H)\Gamma(1-H)}\frac{1}{\delta_N^2} + O(\delta_N^2), \\
I_{H-2}I_{-H} &= \frac{1}{\Gamma(H)\Gamma(1-H)}\frac{4(H-1)}{\delta_N^2} + O(\delta_N^2), \\
I_{-H+2}I_H &= O(\delta_N^2), \quad I_{H+1}I_{-H+1} = O(\delta_N^2),
\end{aligned}$$

we can prove (22) when $\delta_N > 0$. The case of $\delta < 0$ can be proved similarly, which establishes Theorem 4.

Proof of Theorem 5: The relation in (25) can be proved because of the LLN in (23) and the CLT in (24). It follows that

$$\sqrt{N}T\sqrt{\beta_H}\tilde{\alpha}(N,T) \Rightarrow \mathrm{N}(\sqrt{\beta_H}\delta, 1),$$

which yileds (26). Thus Theorem 5 is established.

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