Macroscopic Irreversibility in Quantum Systems

ETH and Equilibration in a Free Fermion Chain

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☑ we prove the presence of macroscopic irreversible behavior (ballistic diffusion) in a free fermion chain initially in a non-random state and evolving under quantum mechanical unitary time-evolution

☑ the proof is based on an accumulation of ideas and methods (in particular, ETH = energy eigenstate thermalization hypothesis) developed to understand thermalization in isolated macroscopic quantum systems, as well as new results specific to the free fermion chain

illustration of Humpty Dumpty from "Through the Looking Glass", by John Tenniel, 1871

introduction/motivation

model and the main theorem

ingredients of the proof

the emergence of macroscopic irreversibility

a physical system governed by a deterministic reversible time-evolution law may exhibit irreversible behavior

it is essential that the system has a large degree of freedom

even an ideal gas may exhibit irreversible behavior

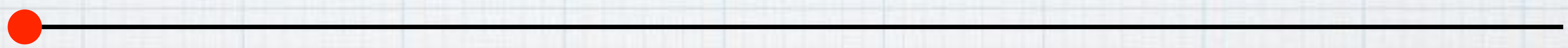
irreversible expansion in a classical ideal gas

N free classical particles on the interval $[0,L]$ with periodic boundary conditions ($N \gg 1$)

$t = 0$

$x_j(0) = 0$ for all j

v_j drawn randomly and uniformly from $[-v_0, v_0]$



$t \gg L/v_0$

$x_j(t) = v_j t \mod L$ are almost uniformly distributed in the interval $[0,L]$

irreversible expansion (or "ballistic diffusion")

the initial velocities must be chosen randomly

classical irreversibility vs quantum irreversibility

classical systems

macroscopic irreversibility can be proved as a probabilistic statement that is valid for the majority of random initial conditions

there are always exceptional initial conditions that does not lead to irreversibility

quantum systems

macroscopic irreversibility can be proved without introducing randomness (either in the Hamiltonian or in the initial state)

we provide a simple rigorous example

introduction/motivation

model and the main theorem

ingredients of the proof

model and non-degeneracy

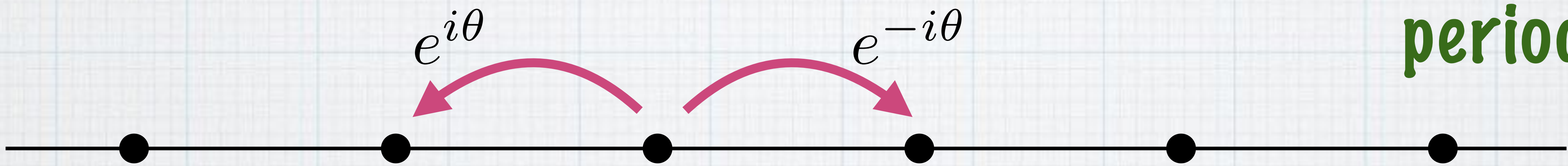
N non-interacting fermions on the chain $\{1,\ldots,L\}$

L a large prime

N a large positive integer

Hamiltonian $\hat{H}=\sum_{x=1}^{L}\{e^{i\theta}\hat{c}_x^\dagger\hat{c}_{x+1}+e^{-i\theta}\hat{c}_{x+1}^\dagger\hat{c}_x\}$ $\theta\in[0,2\pi)$

periodic b.c.



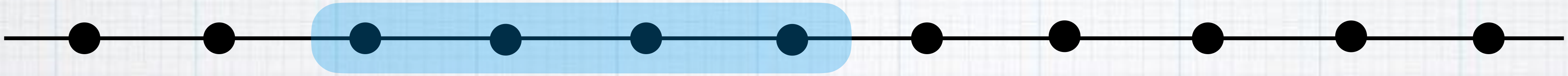
standard free fermion Hamiltonian with nearest neighbor hopping except for the phase factor $e^{i\theta}$

Lemma: all energy eigenvalues of $\hat{H}$ are non-degenerate except for a finite number of θ, in particular, for any $\theta\neq 0$ such that $|\theta|\le(4N+2L)^{-(L-1)/2}$

Tasaki 2010, 2016, Shiraishi, Tasaki 2023

we choose such θ, e.g. $\theta=(4N+2L)^{-(L-1)/2}$

main theorem



S an arbitrary subset of $\{1,\ldots,L\}$ with $|S|$ sites

$\hat{N}_S = \sum_{x\in S} \hat{n}_x$ the number of particles in S

$\mu = \dfrac{|S|}{L}$ the equilibrium value of $\dfrac{\hat{N}_S}{N}$

$\delta > 0$ an arbitrary (sufficiently small) precision

$$|\Phi(t)\rangle = e^{-i\hat{H}t}|\Phi(0)\rangle$$

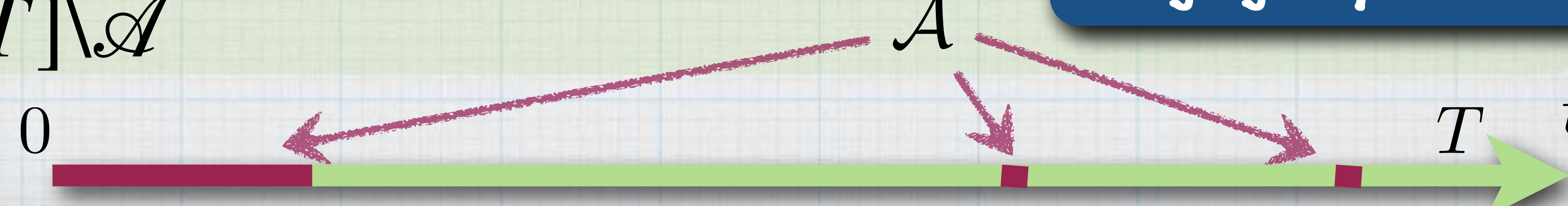
the total length of $\mathscr{A}$

Theorem: for any $|\Phi(0)\rangle$ and any (sufficiently large) T, there exists a set $\mathscr{A} \in [0,T]$ with $\ell(\mathscr{A})/T \le e^{-\frac{\delta^2}{8\mu(1-\mu)}N}$ s.t.

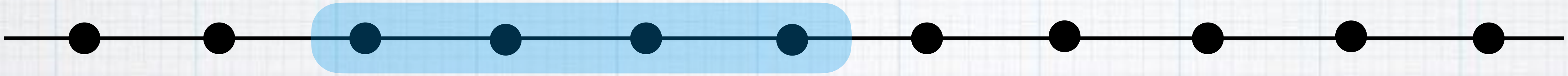
$$\langle\Phi(t)|\hat{P}\left[\left|\frac{\hat{N}_S}{N}-\mu\right| \ge \delta\right]|\Phi(t)\rangle \le e^{-\frac{\delta^2}{8\mu(1-\mu)}N}$$

for any $t \in [0,T]\backslash\mathscr{A}$

negligibly small



main theorem (less formal)



S an arbitrary subset of $\{1,\ldots,L\}$ with $|S|$ sites

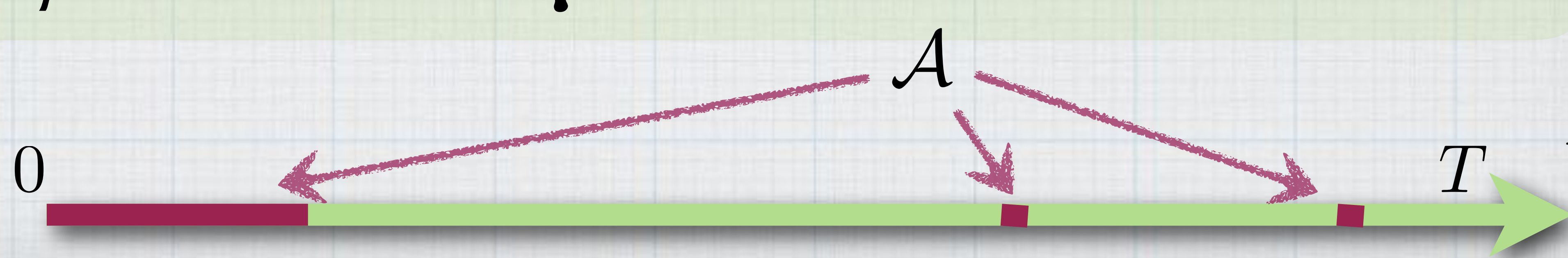
$\hat{N}_S = \sum_{x\in S} \hat{n}_x$ the number of particles in S

$\mu = \dfrac{|S|}{L}$ the equilibrium value of $\dfrac{\hat{N}_S}{N}$

$\delta > 0$ an arbitrary (sufficiently small) precision

$$|\Phi(t)\rangle = e^{-i\hat{H}t}|\Phi(0)\rangle$$

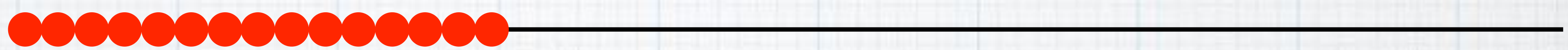
Theorem: let $|\Phi(0)\rangle$ be an arbitrary initial state.
for a sufficiently large and typical time t, the measurement result of $\dfrac{\hat{N}_S}{N}$ in $|\Phi(t)\rangle$ almost certainly equals the equilibrium value μ (within the precision δ).



irreversible expansion

$|\Phi(0)\rangle$ any initial state where all particles are in S

$\frac{\hat{N}_S}{N} = 1$



$|\Phi(t)\rangle$ for sufficiently large and typical t

$\frac{\hat{N}_S}{N} \simeq \mu$

"time's arrow" has emerged from the unitary time evolution in an isolated macroscopic quantum system!
we don't have to introduce randomness in the initial state or the Hamiltonian

it is essential that we focus on a macroscopic observable $\frac{\hat{N}_S}{N}$

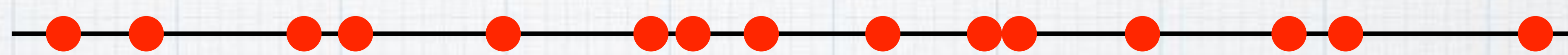
time-reversal "paradox"

$|\Phi(0)\rangle$ any initial state where all particles are in S

$$\frac{\hat{N}_S}{N} = 1$$

$|\Phi(t)\rangle$ for sufficiently large and typical t

$$\frac{\hat{N}_S}{N} \simeq \mu$$



new initial state $|\Xi(0)\rangle = |\Phi(T_0)\rangle^*$ for a typical T_0

$$\frac{\hat{N}_S}{N} \simeq \mu$$

paradox!?

time-evolved state $|\Xi(T_0)\rangle = e^{-i\hat{H}T_0}|\Xi(0)\rangle = |\Phi(0)\rangle^*$

$$\frac{\hat{N}_S}{N} = 1$$

T_0 is not typical with respect to $|\Xi(0)\rangle$

main theorem → for sufficiently large and typical t,

it holds for $|\Xi(t)\rangle$ that $\frac{\hat{N}_S}{N} \simeq \mu$

introduction/motivation

model and the main theorem

ingredients of the proof

strong ETH bound

Hamiltonian $\hat{H} = \sum_{x=1}^{L}\{e^{i\theta}\hat{c}_x^\dagger\hat{c}_{x+1} + e^{-i\theta}\hat{c}_{x+1}^\dagger\hat{c}_x\}$

energy eigenstates

$$|\Psi_{\boldsymbol{k}}\rangle = \hat{a}_{k_1}^\dagger \cdots \hat{a}_{k_N}^\dagger|0\rangle$$

$$\hat{a}_k^\dagger = L^{-1/2}\sum_{x=1}^{L} e^{ikx}\,\hat{c}_x^\dagger \qquad \boldsymbol{k} = (k_1, \ldots, k_N)$$

essential technical result in the present work

Lemma: for every energy eigenstate $|\Psi_{\boldsymbol{k}}\rangle$, we have

$$\langle\Psi_{\boldsymbol{k}}|\hat{P}\Big[\Big|\frac{\hat{N}_S}{N} - \mu\Big| \ge \delta\Big]|\Psi_{\boldsymbol{k}}\rangle \le 2\,e^{-\frac{\delta^2}{3\mu(1-\mu)}N}$$

$\frac{\hat{N}_S}{N}$ almost certainly equals μ (within the precision δ) in every energy eigenstate

strong ETH (energy eigenstate thermalization hypothesis) in the form of large-deviation

von Neumann 1929, Deutsch 1991, Srednicki 1994, Goldstein, Lebowitz, Mastrodonato, Tumulka, and Zanghi 2010, Tasaki 2016

"ergodicity" theorem

initial state $|\Phi(0)\rangle = \sum_{\boldsymbol{k}} \alpha_{\boldsymbol{k}} |\Psi_{\boldsymbol{k}}\rangle$

time-evolved state $|\Phi(t)\rangle = \sum_{\boldsymbol{k}} \alpha_{\boldsymbol{k}}\, e^{-iE_{\boldsymbol{k}}t} |\Psi_{\boldsymbol{k}}\rangle$

expectation value

$$\langle\Phi(t)|\hat{P}|\Phi(t)\rangle = \sum_{\boldsymbol{k},\boldsymbol{k}'} \alpha^*_{\boldsymbol{k}} \alpha_{\boldsymbol{k}'}\, e^{i(E_{\boldsymbol{k}}-E_{\boldsymbol{k}'})t} \langle\Psi_{\boldsymbol{k}}|\hat{P}|\Psi_{\boldsymbol{k}'}\rangle$$

Lemma: all energy eigenvalues of $\hat{H}$ are non-degenerate

long-time average

$$\lim_{T\uparrow\infty} T^{-1} \int_0^T dt\, \langle\Phi(t)|\hat{P}|\Phi(t)\rangle = \sum_{\boldsymbol{k}} |\alpha_{\boldsymbol{k}}|^2 \langle\Psi_{\boldsymbol{k}}|\hat{P}|\Psi_{\boldsymbol{k}}\rangle$$

Lemma: for every energy eigenstate $|\Psi_{\boldsymbol{k}}\rangle$, we have

$$\langle\Psi_{\boldsymbol{k}}|\hat{P}\big[\big|\tfrac{\hat{N}_S}{N} - \mu\big| \ge \delta\big]|\Psi_{\boldsymbol{k}}\rangle \le 2\, e^{-\frac{\delta^2}{3\mu(1-\mu)}N}$$

Theorem: for an arbitrary initial state $|\Phi(0)\rangle$, we have

$$\lim_{T\uparrow\infty} T^{-1} \int_0^T dt\, \langle\Phi(t)|\hat{P}\big[\big|\tfrac{\hat{N}_S}{N} - \mu\big| \ge \delta\big]|\Phi(t)\rangle \le 2\, e^{-\frac{\delta^2}{3\mu(1-\mu)}N}$$

the main theorem is a simple corollary

summary

- ☑ we proved the presence of macroscopic irreversible behavior (ballistic diffusion) in a free fermion chain initially in a non-random state and evolving under quantum mechanical unitary time-evolution
- ☑ the proof is based on an accumulation of ideas and methods (in particular, ETH) developed to understand thermalization in isolated macroscopic quantum systems, as well as new results specific to the free fermion chain

remaining issues

- ☑ extensions to other observables, time scale for equilibration, treatment of non-integrable systems, ...