Two number-theoretic theorems
(that we found useful in quantum physics)
and their elementary proofs

Hal Tasaki

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Nature abhors a vacuum

A simple rigorous example of thermalization in an isolated macroscopic quantum system

Naoto Shiraishi* and Hal Tasaki†

We show, without relying on any unproven assumptions, that a low-density free fermion chain exhibits thermalization in the following (restricted) sense. We choose the initial state as a pure state drawn randomly from the Hilbert space in which all particles are in half of the chain. This represents a nonequilibrium state such that the half chain containing all particles is in equilibrium at infinite temperature and the other half chain is a vacuum. We let the system evolve according to the unitary time evolution determined by the Hamiltonian and, at a sufficiently large typical time, measure the particle number in an arbitrary macroscopic region in the chain. In this setup, it is proved that the measured number is close to the equilibrium value with probability very close to one. Our result establishes the presence of thermalization in a concrete model in a mathematically rigorous manner. The most important theoretical ingredient for the proof of thermalization is the demonstration that a nonequilibrium initial state generated as above typically has a sufficiently large effective dimension. Here, we first give general proof of thermalization based on two assumptions, namely, the absence of degeneracy in energy eigenvalues and a property about the particle distribution in energy eigenstates. We then justify these assumptions in a concrete free-fermion model, where the absence of degeneracy is established by using number-theoretic results. This means that our general result also applies to any lattice gas models in which the above two assumptions are justified. To confirm the potential wide applicability of our theory, we discuss some other models for which the essential assumption about the particle distribution is easily verified, and some non-random initial states whose effective dimensions are sufficiently large.

Lemma 3.3 *For any $m_1, \ldots, m_{L-1} \in \mathbb{Z}$ such that $m_\mu \neq 0$ for some μ, one has*

$$\sum_{\mu=1}^{L-1} m_\mu \zeta^\mu \neq 0. \tag{3.16}$$

The lemma is a straightforward consequence of the classical result by Gauss, known as the irreducibility of the cyclotomic polynomials of prime index. See, e.g., Chapter 12, Section 2 of [47] or Chapter 13, Section 2 of [48].

The following lemma[5] provides an explicit lower bound for $|\sum_{\mu=1}^{L-1} m_\mu \zeta^\mu|$.

Lemma 3.4 *For any $m_1, \ldots, m_{L-1} \in \mathbb{Z}$ such that $\sum_{\mu=1}^{L-1} |m_\mu| = M > 0$, one has*

$$\left|\sum_{\mu=1}^{L-1} m_\mu \zeta^\mu\right| \geq \frac{1}{M^{(L-3)/2}}. \tag{3.17}$$

many thanks to Shin Nakano

main references

I. Stewart and D. Tall, Algebraic number theory and Fermat's last theorem

S. Nakano's lecture notes (in Japanese)

J. Nakagawa's lecture notes (in Japanese)

⟨main theorems⟩

p odd prime $\qquad \boxed{\zeta = e^{i\frac{2\pi}{p}}} \rightarrow \zeta^p = 1$

Theorem I: for any $c_1, \dots, c_{p-1} \in \mathbb{Q}$ with $c_n \neq 0$ for some n

$$\sum_{n=1}^{p-1} c_n \zeta^n \neq 0$$

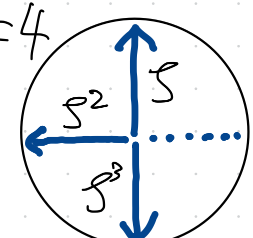
well-known theorem that follows from famous theorem by Gauss

$\zeta, \zeta^2, \dots, \zeta^{p-1}$ are linearly independent (when $c_1, \dots, c_{p-1} \in \mathbb{Q}$)

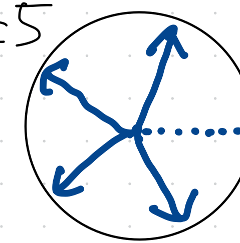
remarks (1) it is essential that p is a prime

(2) one never has linear independence if $c_1, \dots, c_{p-1} \in \mathbb{R}$ (and $p > 3$)

$p=4$



$p=5$



Corollary: for any $m_1, \dots, m_{p-1} \in \mathbb{Z}$ with $m_n \neq 0$ for some n, $\sum_{n=1}^{p-1} m_n \zeta^n \neq 0$

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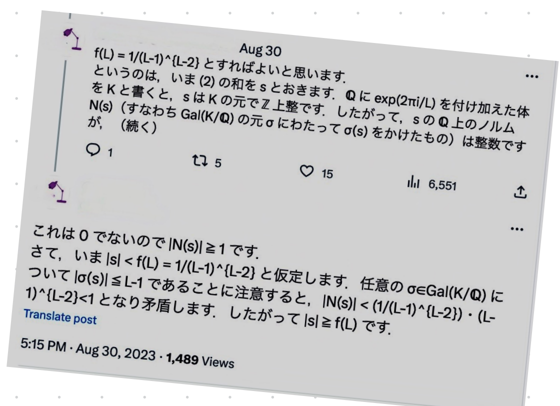
Theorem II: for any $m_1, \dots, m_{p-1} \in \mathbb{Z}$ with $m_n \neq 0$ for some n

$$\left| \sum_{n=1}^{p-1} m_n \zeta^n \right| \geq \left(\sum_{n=1}^{p-1} |m_n| \right)^{-\frac{p-3}{2}}$$

W. Kai, K. Miyatani



Aug 30

f(L) = 1/(L-1)^{L-2} とすればよいと思います.
というのは，いま (2) の和を s とおきます．ℚ に exp(2πi/L) を付け加えた体を K と書くと，s は K の元で ℤ 上整です．したがって，s の ℚ 上のノルム N(s)（すなわち Gal(K/ℚ) の元 σ にわたって σ(s) をかけたもの）は整数ですが，（続く）

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これは 0 でないので |N(s)| ≧ 1 です．
さて，いま |s| < f(L) = 1/(L-1)^{L-2} と仮定します．任意の σ∈Gal(K/ℚ) について |σ(s)| ≦ L-1 であることに注意すると，|N(s)| < (1/(L-1)^{L-2})・(L-1)^{L-2}<1 となり矛盾します．したがって |s| ≧ f(L) です．

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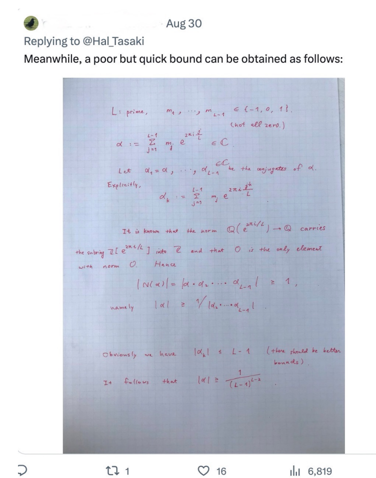

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Replying to @Hal_Tasaki

Meanwhile, a poor but quick bound can be obtained as follows:

L: prime, $m_1, \dots, m_{L-1} \in \{-1, 0, 1\}$. (not all zero.)

$\alpha := \sum_{j=1}^{L-1} m_j e^{2\pi i \frac{j}{L}} \in \mathbb{C}$.

Let $\alpha_1 = \alpha, \dots, \alpha_{L-1} \in \mathbb{C}$ be the conjugates of α.

Explicitly, $\alpha_k := \sum_{j=1}^{L-1} m_j e^{2\pi i \frac{jk}{L}}$.

It is known that the norm $\mathbb{Q}(e^{2\pi i/L}) \to \mathbb{Q}$ carries the subring $\mathbb{Z}[e^{2\pi i/L}]$ into $\mathbb{Z}$ and that 0 is the only element with norm 0. Hence

$|N(\alpha)| = |\alpha \cdot \alpha_2 \cdot \dots \cdot \alpha_{L-1}| \geq 1$,

namely $|\alpha| \geq 1 / |\alpha_2 \cdot \dots \cdot \alpha_{L-1}|$.

Obviously we have $|\alpha_k| \leq L-1$ (there should be better bounds).

It follows that $|\alpha| \geq \frac{1}{(L-1)^{L-2}}$.

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<proof of Theorem I>

Theorem I: for any $c_1, \dots, c_{p-1} \in \mathbb{Q}$ with $c_n \neq 0$ for some n
$$\sum_{n=1}^{p-1} c_n \zeta^n \neq 0$$

a formal polynomial in t

$$t^p - 1 = (t-1) f(t) \quad \text{with} \quad f(t) = t^{p-1} + t^{p-2} + \cdots + 1$$

cyclotomic polynomial

$$\zeta^p - 1 = 0 \text{ and } \zeta \neq 1 \implies f(\zeta) = 0$$

Gauss

Lemma: $f(t)$ is irreducible
(cannot be factorized within polynomials over $\mathbb{Q}$)

$\Updownarrow$

we never have $f(t) = \left(\sum_{n=0}^{q} b_n t^n\right)\left(\sum_{m=0}^{r} c_m t^m\right)$ with $q, r \geq 1$
$b_0, \dots, b_q, c_0, \dots, c_r \in \mathbb{Q}$

proof of Lemma

$$\circledast \quad f(t) = \left(\sum_{n=0}^{q} b_n t^n\right)\left(\sum_{m=0}^{r} c_m t^m\right) \quad \text{with } q, r \geq 1$$

it suffices to show we never have $\circledast$ for $b_0, \cdots, b_q, c_0, \cdots, c_r \in \mathbb{Z}$.

suppose $\circledast$ with $b_0, \cdots, b_q, c_0, \cdots, c_r \in \mathbb{Q}$

then $\exists N \in \mathbb{Z}$

$$N f(t) = \left(\sum_{n=0}^{q} B_n t^n\right)\left(\sum_{m=0}^{r} C_m t^m\right)$$

with $B_0, \cdots, B_q, C_0, \cdots, C_r \in \mathbb{Z}$.

but

$$f(t) = \frac{\left(\sum_{n=0}^{q} B_n t^n\right)\left(\sum_{m=0}^{r} C_m t^m\right)}{p_1 \cdots p_\ell} \qquad p_1 \cdots p_\ell = N \qquad p_1, \cdots, p_\ell \text{ prime}$$

one finds p_j devides all $B_0, \cdots, B_q$ or all $C_0, \cdots, C_r$

so we have $\circledast$ with $b_0, \cdots, b_q, c_0, \cdots, c_r \in \mathbb{Z}$

$$f(t) = \frac{t^p - 1}{t - 1} \qquad f(s+1) = \frac{(s+1)^p - 1}{s} = \sum_{n=0}^{p-1} a_n s^n \text{ with } a_n = \binom{p}{n+1}$$

$$a_{p-1} = \binom{p}{p} = 1, \quad a_0 = \binom{p}{1} = p, \quad a_0, \cdots, a_{p-2} \text{ devisible by } p$$

assume $f(s+1) = \overbrace{s^{p-1} + ps^{p-2} + \cdots + \frac{p(p-1)}{2} s + p}^{\sum_{n=0}^{p-1} a_n s^n}$

is factorized as $= (\underbrace{s^q + \cdots + p}_{\sum_{n=0}^{q} b_n s^n})(\underbrace{s^r + \cdots + 1}_{\sum_{m=0}^{r} c_m s^m})$ with $b_n, c_m \in \mathbb{Z}$

$q, r \geq 1, \; q + r = p - 1$

$\ell \leq p-2$

let ℓ $(1 \leq \ell \leq q)$ be s.t. $\underset{p}{b_0}, \cdots, b_{\ell-1}$ devisible by p, but b_ℓ is not

then $a_\ell = \sum_{\substack{n,m \\ (n+m=\ell)}} b_n c_m = \underbrace{b_\ell \overset{1}{c_0}}_{\text{not devisible by } p} + \underbrace{b_{\ell-1} c_1 + \cdots + b_0 c_\ell}_{\text{devisible by } p}$

contradiction! (Eisenstein's criterion) //

Theorem I: for any $c_1, \ldots, c_{p-1} \in \mathbb{Q}$ with $c_n \neq 0$ for some n

$$\sum_{n=1}^{p-1} c_n \zeta^n \neq 0$$

proof let $g(t) = \sum_{n=1}^{p-1} c_n t^{n-1}$

$p-1$ $\quad$ p

from irreducibility of $f(t)$ and $\deg(g) < \deg(f)$, one can prove that $\exists$ polynomials (over $\mathbb{Q}$) $a(t), b(t)$ s.t.

$$(\star) \quad a(t) f(t) + b(t) g(t) = 1$$

letting $t = \zeta$ and recalling $f(\zeta) = 0$, we have

$$b(\zeta) g(\zeta) = 1 \quad \text{and hence} \quad g(\zeta) \neq 0$$

$$\therefore \sum_{n=1}^{p-1} c_n \zeta^n = \zeta g(\zeta) \neq 0$$

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proof of ⊛ Euclidean algorithm for polynomials over $\mathbb{Q}$

write $f_1 = f$, $f_2 = g$ $\quad f(t) = t^{p-1} + \cdots + 1$, $g(t) = \sum_{n=0}^{p-2} c_{n+1} t^n$

devide f_1 by f_2 → $\boxed{f_1 = q_1 f_2 + f_3}$ = remainder $\quad \deg(f_1) > \deg(f_2) > \deg(f_3)$

d is a common factor of f_1 and f_2

⇕

d is a common factor of f_2 and f_3

$\boxed{f_j = q_j f_{j+1} + f_{j+2}}$ $(j = 1, 2, \cdots, k-2)$ $\quad \deg(f_j) > \deg(f_{j+1})$

$f_{k-1} = q_{k-1} f_k + 0$ = remainder

f_k a common factor of f_{k-1} and f_k

a factor of $f_1 = f$ ⟹ $f_k = c \in \mathbb{Q} \setminus \{0\}$

irreducible!!

$$\boxed{f_j = q_j f_{j+1} + f_{j+2}} \quad (j=1,2,\cdots,k-2)$$

$$f_j = f_{j-2} - q_{j-2} f_{j-1} \quad (j=3,\cdots,k)$$

$$\therefore \quad f_k = f_{k-2} - q_{k-2} f_{k-1}$$

$$= f_{k-4} - q_{k-4} f_{k-3} - q_{k-2}\{f_{k-3} - q_{k-3} f_{k-2}\}$$

$$\vdots$$

$$= \tilde{a} f_1 + \tilde{b} f_2$$

written in terms of $q_1, \cdots, q_{k-1}$

since $f_k = c$ we have $\tilde{a}(t)\underbrace{f_1(t)}_{= f(t)} + \tilde{b}(t)\underbrace{f_2(t)}_{= g(t)} = c$

$$a(t)f(t) + b(t)g(t) = 1$$

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example (of the proof of $\circledast$)

$$f = f_1 = t^4+t^3+t^2+t+1,\quad g = f_2 = t^3+\frac{1}{2}$$

$$f_1 = \underbrace{\boxed{(t+1)}}_{q_1 \,=} f_2 + \underbrace{\boxed{t^2+\frac{t}{2}+\frac{1}{2}}}_{=f_3}$$

$$f_2 = \underbrace{\boxed{\left(t-\frac{1}{2}\right)}}_{q_2\,=} f_3 \underbrace{\boxed{-\frac{t}{4}+\frac{3}{4}}}_{=f_4}$$

$$f_3 = \underbrace{\boxed{(-4t-14)}}_{q_3\,=} f_4 + \underbrace{\boxed{11}}_{=f_5}$$

$$\begin{array}{r l}
 & \qquad\qquad\qquad\quad t+1 \\
t^3+\frac{1}{2} \;\big) & t^4+t^3+t^2+t+1 \\
 & t^4 \qquad\qquad\;\; +\frac{t}{2} \\
\hline
 & \quad t^3+t^2+\frac{t}{2}+1 \\
 & \quad t^3 \qquad\qquad +\frac{1}{2} \\
\hline
 & \qquad t^2+\frac{t}{2}+\frac{1}{2}
\end{array}$$

now

$$f_5 = f_3 - q_3 f_4 = \cdots = (1+q_2q_3)f_1 - (q_1+q_3+q_1q_2q_3)f_2$$

$$\therefore\; af+bg = 1$$

with $a = \dfrac{1}{f_5}(1+q_2q_3) = -\dfrac{4}{11}(t^2+3t-2)$

$b = \dfrac{1}{f_5}(q_1+q_3+q_1q_2q_3) = \dfrac{2}{11}(2t^3+8t^2+4t+3)$

<proof of Theorem 2>

$$\mathbb{Q}[\zeta] = \left\{ \sum_{n=1}^{p-1} c_n \zeta^n \,\middle|\, c_1, \cdots, c_{p-1} \in \mathbb{Q} \right\}, \quad \mathbb{Z}[\zeta] = \left\{ \sum_{n=1}^{p-1} m_n \zeta^n \,\middle|\, m_1, \cdots, m_{p-1} \in \mathbb{Z} \right\}$$

▷ Conjugates $\quad j = 1, \cdots, p-1$

$(\zeta^j, \zeta^{2j}, \cdots, \zeta^{(p-1)j})$ is a permutation of $(\zeta, \zeta^2, \cdots, \zeta^{p-1})$

$\left(\because \ 1 \le n < n' \le p-1 \Rightarrow nj \not\equiv n'j \bmod p \right)$

example $p = 5$ $\quad (\zeta^3, \zeta^6, \zeta^9, \zeta^{12}) = (\zeta^3, \zeta, \zeta^4, \zeta^2)$

$$\alpha = \sum_{n=1}^{p-1} c_n \zeta^n \in \mathbb{Q}[\zeta]$$

conjugate $\boxed{\sigma_j(\alpha) = \sum_{n=1}^{p-1} c_n \zeta^{nj}} \in \mathbb{Q}[\zeta]$

by definition $\sigma_j(\alpha + \beta) = \sigma_j(\alpha) + \sigma_j(\beta)$, $\quad \sigma_j(\alpha\beta) = \sigma_j(\alpha)\sigma_j(\beta)$

14 field norm

for $\alpha \in \mathbb{Q}[\zeta]$

define its norm by $\boxed{N(\alpha) = \prod_{j=1}^{p-1} \sigma_j(\alpha)}$

$\boxed{\text{Lemma} \quad N(\alpha) \in \mathbb{Z} \text{ if } \alpha \in \mathbb{Z}[\zeta]}$

proof for experts (not me)

generally $N(\alpha) \in \mathbb{Q}$ (for $\alpha \in \mathbb{Q}[\zeta]$)

$\alpha \in \mathbb{Z}[\zeta]$ is an algebraic integer, and so are $\sigma_j(\alpha)$, $N(\alpha)$.

an algebraic integer that is in $\mathbb{Q}$ is in $\mathbb{Z}$ //

We shall give a (standard) elementary proof at the end.

Theorem II: for any $m_1, \ldots, m_{p-1} \in \mathbb{Z}$ with $m_n \neq 0$ for some n

$$\left| \sum_{n=1}^{p-1} m_n \zeta^n \right| \geq \left(\sum_{n=1}^{p-1} |m_n| \right)^{-(p-3)/2}$$

proof of Theorem II, given Lemma

$$\alpha = \sum_{n=1}^{p-1} m_n \zeta^n \in \mathbb{Z}[\zeta], \quad \text{conjugate } \sigma_j(\alpha) = \sum_{n=1}^{p-1} m_n \zeta^{nj}$$

note that $\overline{\sigma_j(\alpha)} = \sum_{n=1}^{p-1} m_n \zeta^{-nj} = \sum_{n=1}^{p-1} m_n \zeta^{n(p-j)} = \sigma_{p-j}(\alpha)$

$\sigma_1(\alpha) = \alpha$.

$$\therefore N(\alpha) = \prod_{j=1}^{p-1} \sigma_j(\alpha) = \prod_{j=1}^{(p-1)/2} |\sigma_j(\alpha)|^2 = |\alpha|^2 \prod_{j=2}^{(p-1)/2} |\sigma_j(\alpha)|^2 \geq 0$$

$$\text{Theorem I} \Rightarrow \sigma_j(\alpha) = \sum_{n=1}^{p-1} m_n \zeta^{nj} \neq 0$$

$$\Rightarrow N(\alpha) = \prod_{j=1}^{p-1} \sigma_j(\alpha) > 0$$

but $N(\alpha) \in \mathbb{Z}$ from Lemma. $\Rightarrow \boxed{N(\alpha) \geq 1}$

$\therefore$

$$N(\alpha) = |\alpha|^2 \prod_{j=2}^{(p-1)/2} |\sigma_j(\alpha)|^2 \geq 1$$

$$|\alpha|^2 \geq \left(\prod_{j=2}^{(p-1)/2} |\sigma_j(\alpha)|^2 \right)^{-1} \geq \left(\sum_{n=1}^{p-1} |m_n| \right)^{-\frac{p-3}{2}}$$

$$\boxed{|\sigma_j(\alpha)| = \left| \sum_{n=1}^{p-1} m_n \zeta^{nj} \right| \leq \sum_{n=1}^{p-1} |m_n|}$$

//

proof of Lemma

$\boxed{\text{Lemma } N(\alpha) \in \mathbb{Z} \text{ if } \alpha \in \mathbb{Z}[\zeta]}$

$$\zeta\, \zeta^n = \begin{cases} \zeta^{n+1} & (n=1,\cdots,p-2) \\ \zeta^p = -\sum_{m=1}^{p-1} \zeta^m & (n=p-1) \end{cases}$$

write this as

$$\zeta \begin{pmatrix} \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{p-2} \\ \zeta^{p-1} \end{pmatrix} = \begin{pmatrix} \zeta^2 \\ \zeta^3 \\ \vdots \\ \zeta^{p-1} \\ -\sum_{m=1}^{p-1} \zeta^m \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix}}_{= Z} \begin{pmatrix} \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{p-2} \\ \zeta^{p-1} \end{pmatrix} \qquad (*)$$

clearly

$$\zeta^n \begin{pmatrix} \zeta \\ \vdots \\ \zeta^{p-1} \end{pmatrix} = Z^n \begin{pmatrix} \zeta \\ \vdots \\ \zeta^{p-1} \end{pmatrix} \qquad n=1,2,\cdots,p-1$$

(*)

$$\zeta \begin{pmatrix} \zeta \\ \vdots \\ \zeta^{p-1} \end{pmatrix} = Z \begin{pmatrix} \zeta \\ \vdots \\ \zeta^{p-1} \end{pmatrix}$$

apply σ_j to each entry

$$\underset{\zeta^j}{\underset{\shortparallel}{\sigma_j(\zeta)}} \begin{pmatrix} \sigma_j(\zeta) \\ \vdots \\ \sigma_j(\zeta^{p-1}) \end{pmatrix} = Z \begin{pmatrix} \sigma_j(\zeta) \\ \vdots \\ \sigma_j(\zeta^{p-1}) \end{pmatrix} \qquad j=1,\cdots,p-1$$

$\zeta, \cdots, \zeta^{p-1}$: eigenvalues of Z

$\longrightarrow$ distinct $\longrightarrow$ Z is diagonalizable

$$P^{-1} Z P = \begin{pmatrix} \zeta & & & \\ & \zeta^2 & & \\ & & \ddots & \\ & & & \zeta^{p-1} \end{pmatrix}$$

$$\alpha = \sum_{n=1}^{p-1} c_n \zeta^n \in \mathbb{Q}[\zeta]$$

similarly $\quad \alpha \begin{pmatrix} \zeta \\ \vdots \\ \zeta^{p-1} \end{pmatrix} = A \begin{pmatrix} \zeta \\ \vdots \\ \zeta^{p-1} \end{pmatrix}$

with $\quad A = \sum_{n=1}^{p-1} c_n Z^n$

then $\quad P^{-1} A P = \sum_{n=1}^{p-1} c_n (P^{-1} Z P)^n = \sum_{n=1}^{p-1} c_n \begin{pmatrix} \zeta^n & & & \\ & \zeta^{2n} & & \\ & & \ddots & \\ & & & \zeta^{(p-1)n} \end{pmatrix}$

$$= \begin{pmatrix} \sigma_1(\alpha) & & & \\ & \sigma_2(\alpha) & & \\ & & \ddots & \\ & & & \sigma_{p-1}(\alpha) \end{pmatrix}$$

we then find

$$\det[A] = \det[P^{-1}AP] = \prod_{j=1}^{p-1} \sigma_j(\alpha) = N(\alpha)$$

and hence

$$\alpha \in \mathbb{Q}[\zeta] \Rightarrow (A)_{nm} \in \mathbb{Q} \Rightarrow N(\alpha) = \det[A] \in \mathbb{Q}$$

$$\alpha \in \mathbb{Z}[\zeta] \Rightarrow (A)_{nm} \in \mathbb{Z} \Rightarrow N(\alpha) = \det[A] \in \mathbb{Z}$$

$$A = \sum_{n=1}^{p-1} c_n Z^n \quad , \quad Z = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ \vdots & \vdots & & & \vdots \\ -1 & -1 & -1 & \cdots & -1 \end{pmatrix}$$

<main theorems>

p odd prime $\quad \boxed{\zeta = e^{i\frac{2\pi}{p}}} \to \zeta^p = 1$

Theorem I: for any $c_1, \ldots, c_{p-1} \in \mathbb{Q}$ with $c_n \neq 0$ for some n

$$\sum_{n=1}^{p-1} c_n \zeta^n \neq 0$$

well-known theorem that follows from famous theorem by Gauss

$\zeta, \zeta^2, \ldots, \zeta^{p-1}$ are linearly independent (when $c_1, \ldots, c_{p-1} \in \mathbb{Q}$)

Theorem II: for any $m_1, \ldots, m_{p-1} \in \mathbb{Z}$ with $m_n \neq 0$ for some n

$$\left|\sum_{n=1}^{p-1} m_n \zeta^n\right| \geq \left(\sum_{n=1}^{p-1} |m_n|\right)^{-(p-3)/2}$$