Symmetry-protected topological (SPT) phases and topological indices in quantum spin chains

appendix: Definition of Ogata index (some operator algebra)

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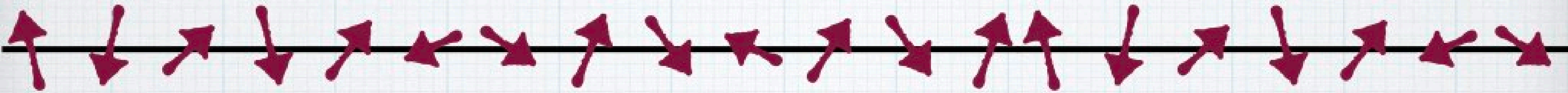
online lecture @ YouTube / August 2021

General quantum spin chain



$\mathfrak{h}_j$ **Hilbert space at site** $j \in \mathbb{Z}$ $\qquad \dim(\mathfrak{h}_j) \le d_0$

C^***-algebra** $\quad \mathfrak{A} = \overline{\{\text{all local operators}\}}$

G **symmetry group (finite group)**

$u_g^{(j)}$ **unitary on** $\mathfrak{h}_j$

projective representation with index $\mathrm{ind}_j \in \mathrm{H}^2(G, \mathrm{U}(1))$

***-automorphism on** $\mathfrak{A}$

$$\Xi_g(A) = \Big(\bigotimes_{j=-L}^{L} u_g^{(j)}\Big) A \Big(\bigotimes_{j=-L}^{L} u_g^{(j)}\Big)^*$$

for $g \in G$ **and a local operator** A

$$\Xi_g \circ \Xi_h = \Xi_{gh}$$

G-invariant Hamiltonian and a unique gapped g.s.

formal expression

G-invariant short ranged Hamiltonian $H = \sum_{j\in\mathbb{Z}} h_j$

$h_j = h_j^*$ acts only on $\bigotimes_{k;|k-j|\le r_0} \mathfrak{h}_k$

$\Xi_g(h_j) = h_j$ for any $j \in \mathbb{Z}$ and $g \in G$

basic assumption: the ground state ω of H is unique and accompanied by a nonzero energy gap

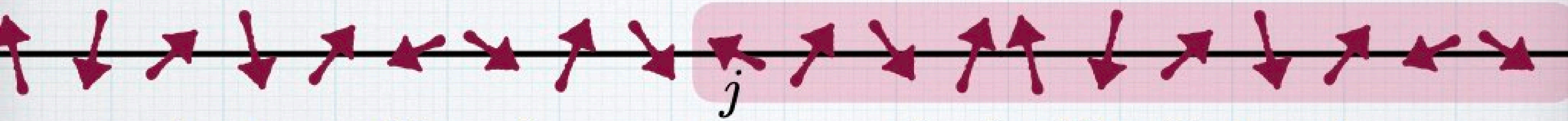
$$\omega(A) = \lim_{L\uparrow\infty} \langle \Phi_{\mathrm{GS}}^{(L)}, A\Phi_{\mathrm{GS}}^{(L)} \rangle$$

Def: a state is a linear function $\omega : \mathfrak{A} \to \mathbb{C}$ such that $\omega(I) = 1$ and $\omega(A^*A) \ge 0$ for any $A \in \mathfrak{A}$

Def: ω is a g.s. if $\omega(A^*[H, A]) \ge 0$ for any local operator A

Def: a unique g.s. ω is accompanied by a nonzero gap if there is $\gamma > 0$ and $\omega(A^*[H, A]) \ge \gamma\, \omega(A^*A)$ for any A s.t. $\omega(A) = 0$

GNS Hilbert space for half infinite chain



C^*-algebra of local operators on the half-infinite chain

$$\mathfrak{A}_j = \overline{\{\text{all local operators on } \{j, j+1, \ldots\}\}} \qquad j \in \mathbb{Z}$$

ω_j restriction of the ground state ω on $\mathfrak{A}_j$

$\mathfrak{A}_j$ and ω_j —GNS construction→ $(\mathcal{H}_j, \pi_j, \Omega_j)$ ($\Omega_j \in \mathcal{H}_j$)

representation $\pi_j : \mathfrak{A}_j \to B(\mathcal{H}_j)$

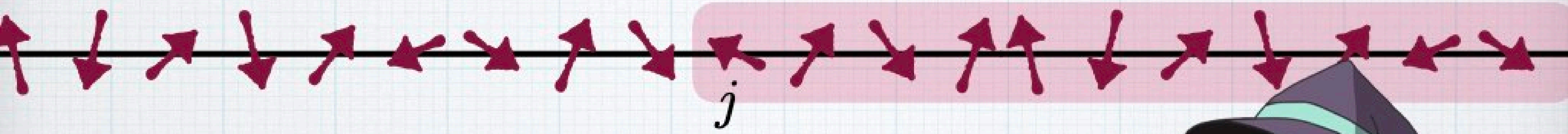
$\omega_j(A) = \langle \Omega_j, \pi_j(A)\, \Omega_j \rangle$ $\quad \{\pi_j(A)\, \Omega_j \mid A \in \mathfrak{A}_j\}$ is dense in $\mathcal{H}_j$

noting the G-invariance $\omega_j(\Xi_g(A)) = \omega_j(A)$, we can define unitary U_g on $\mathcal{H}_j$ by $U_g\, \pi_j(A)\, \Omega_j = \pi_j(\Xi_g(A))\, \Omega_j$ for $A \in \mathfrak{A}_j$

but ... $U_g U_h = U_{gh}$ $\quad$ genuine rep. = trivial proj. rep.

this is not yet what we want!

von Neumann algebra for half infinite chain



$\pi_j(\mathfrak{A}_j)$ —bicommutant→ $\pi_j(\mathfrak{A}_j)''$

representation of the C* algebra

von Neumann algebra

$$\pi_j(\mathfrak{A}_j) \subset \pi_j(\mathfrak{A}_j)'' \subset B(\mathcal{H}_j)$$

← the set of all bounded operators on $\mathcal{H}_j$

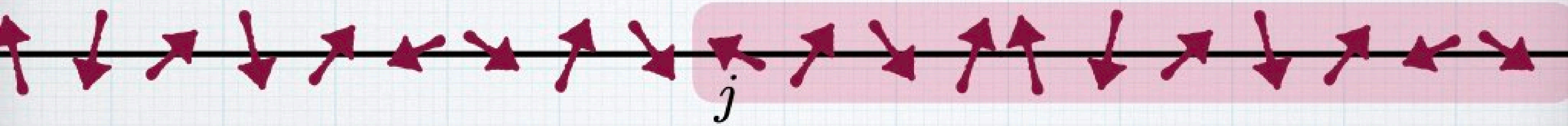
when ω is a unique gapped ground state $\pi_j(\mathfrak{A}_j)''$ is a type-I factor, which is the most well-behaved von Neumann algebra Matsui 2013

then $\pi_j(\mathfrak{A}_j)'' \cong B(\tilde{\mathcal{H}}_j)$ for some Hilbert space $\tilde{\mathcal{H}}_j$

illustration by Chisato Naruse

proj. rep. on half infinite chain

Matsui 2001



$\pi_j(\mathfrak{A}_j)'' \cong B(\tilde{\mathcal{H}}_j)$ for some Hilbert space $\tilde{\mathcal{H}}_j$

one can construct a projective rep. $\tilde{U}_g$ of G on $\tilde{\mathcal{H}}_j$

the corresponding index $\mathrm{Ind}_j \in \mathrm{H}^2(G, \mathrm{U}(1))$

rough idea of the construction

$\pi_j(\mathfrak{A}_j)$ is invariant under the action of $U_g(\cdot)U_g^*$

define $*$-automorphism Γ_g on $B(\tilde{\mathcal{H}}_j)$ by

$$\Gamma_g(X) = \varphi(U_g\,\varphi^{-1}(X)\,U_g^*) \qquad \pi_j(\mathfrak{A}_j)'' \xrightarrow{\varphi} B(\tilde{\mathcal{H}}_j)$$

it holds that $\Gamma_g\Gamma_h = \Gamma_{gh}$

Wigner's theorem guarantees that there is a unitary $\tilde{U}_g$ on $\tilde{\mathcal{H}}_j$ such that $\Gamma_g(X) = \tilde{U}_g\,X\,\tilde{U}_g^*$