統計物理学懇談会@慶応大

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対称性に保護されたトポロジカル(SPT)相と場の理論

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ST, K. Totsuka, and A. Tanaka, Phys. Rev. B **91**, 155136 (2015).
ST, P. Pujol, and A. Tanaka, Phys. Rev. B **94**, 235159 (2016).

Outline

- Introduction

  What is SPT?

- SPT state in 1D antiferromagnets

  AKLT VBS state, Haldane phase, MPS

- Field theory of SPT state

  Nonlinear sigma model, GS wave functional

- Strange correlator

  Indicator for SPT states

- Conclusion

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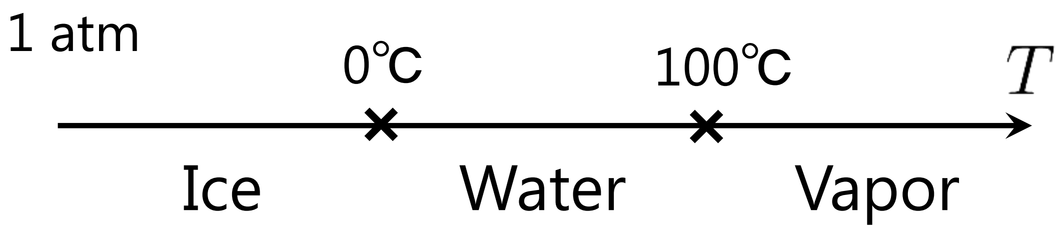
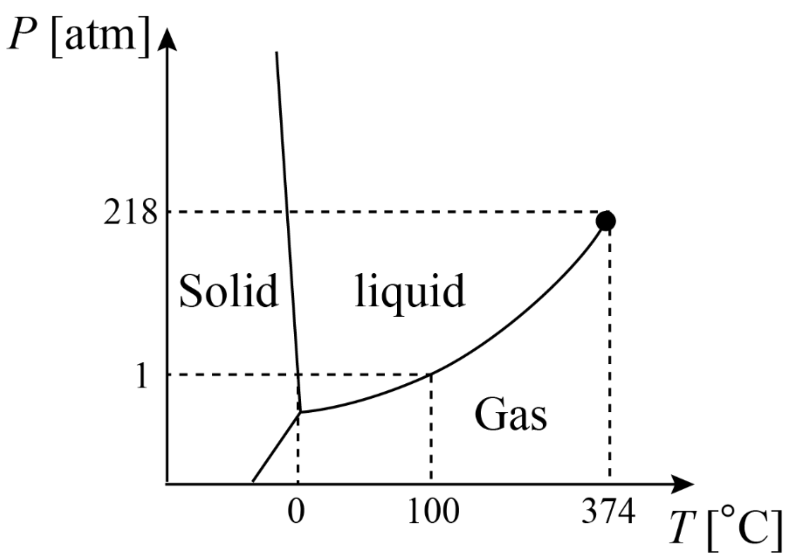
What are different phases?

Phase transition

Ice / Water / Vapor





Water and Vapor are the same phase.

Landau theory

Phase transition

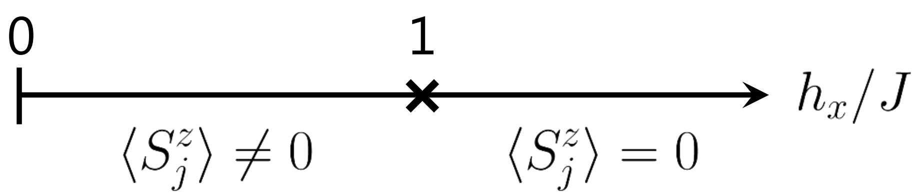
-> spontaneous symmetry breaking.

Ice / (Water, Vapor) : translational symmetry

We can define a local order parameter.

$$\mathcal{H} = -J \sum_j S_j^z S_{j+1}^z - h_x \sum_j S_j^x$$

Transverse Ising model



In this talk, only $T = 0$ is considered.

Same phase: connected with continuous change of parameters in $\mathcal{H}$.

SPT phase/state

Gapped phase

- Long-range entangled phase

  GS ⟶(✗) direct product state with local unitary.
  FQHE, Z_2 spin liquid, etc.

- SSB phase

  Landau theory, local order parameter

- SPT phase

  GS ⟶(✗) direct product state
  only if some symmetry is imposed.

- Trivial phase

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  - AKLT VBS state, Haldane phase
- Field theory of 1D SPT state
  - Nonlinear sigma model, GS wave functional
- 2D or higher spin systems
  - 2D AKLT VBS state, (Group cohomology)
- Conclusion

Integer spin antiferromagnets

Heisenberg model $\mathcal{H} = J\sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \ (J > 0)$

Gapped, No SSB for integer spin

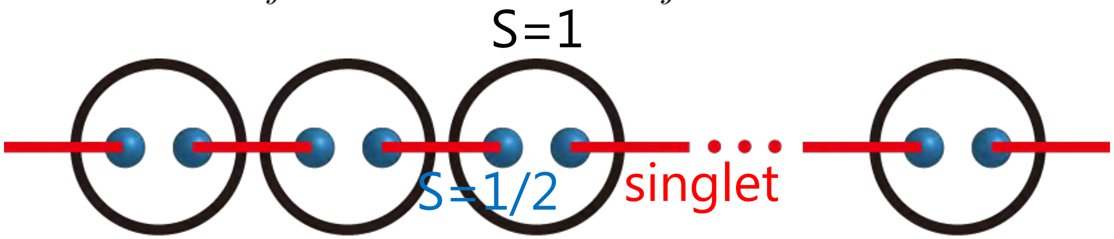
F. D. M. Haldane, Phys. Lett. A **93**, 464 (1983);
Phys. Rev. Lett. **50**, 1153 (1983).

AKLT VBS state

I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987); Commun. Math. Phys. **115**, 477 (1988).

S=1

$$\mathcal{H} = J\Big[\sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}\sum_j (\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 + \frac{2}{3}\Big]$$

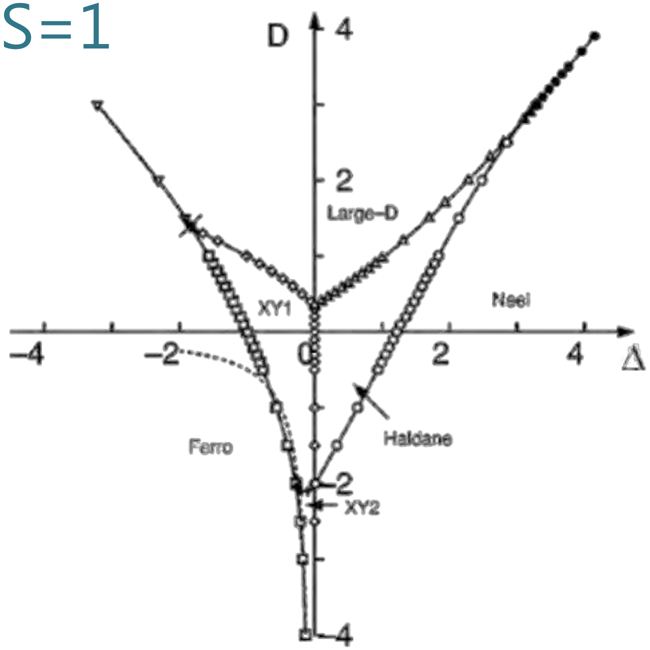


1D antiferromagnets

$$\mathcal{H} = J\sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) + D\sum_j (S_j^z)^2$$

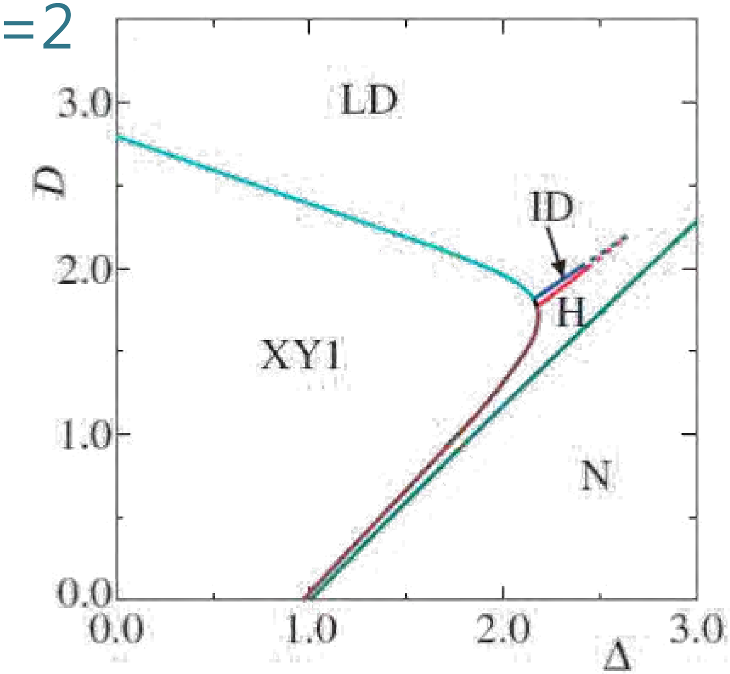
Large-*D* state (direct product) $|000\cdots\rangle$

S=1



Chen et al., (2003)

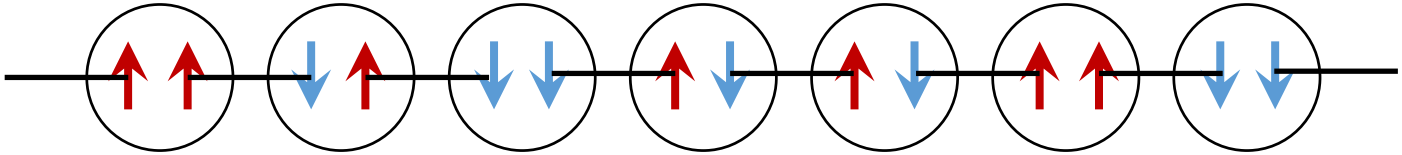
S=2



Tonegawa et al., (2011)

String order parameter

M. den Nijs and K. Rommelse, Phys. Rev. B **40**, 4709 (1989)



Néel order without 0

$$C_{\mathrm{str}}(j,k) \equiv \langle S_j^\alpha \prod_{l=j}^{k-1} e^{i\pi S_l^\alpha} S_k^\alpha \rangle$$

$$e^{i\pi S_l^\alpha} = \begin{cases} +1 & (S_l^\alpha = 0) \\ -1 & (S_l^\alpha = \pm 1) \end{cases}$$

$\lim_{|j-k|\to\infty} C_{\mathrm{str}}(j,k)$: String order parameter

Hidden $Z_2 \times Z_2$ symmetry breaking

T. Kennedy and H. Tasaki, PRB **45**, 304 (1992)

Nonlocal unitary transformation for o.b.c.

$$U = \prod_{j<k} e^{S_j^z S_k^x}$$

For general-S,
M. Oshikawa, J. Phys.: Cond. Mat. **4**, 7469 (1992)

$$\mathcal{H} = J \sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) + D \sum_j (S_j^z)^2$$

$$\tilde{\mathcal{H}} = U \mathcal{H} U^{-1}$$
$$= J \sum_j (S_j^x e^{i\pi S_{j+1}^x} S_{j+1}^x + S_j^y e^{i\pi (S_j^z + S_{j+1}^x)} S_{j+1}^y + \Delta S_j^z e^{i\pi S_j^z} S_{j+1}^z) + D \sum_j (S_j^z)^2$$

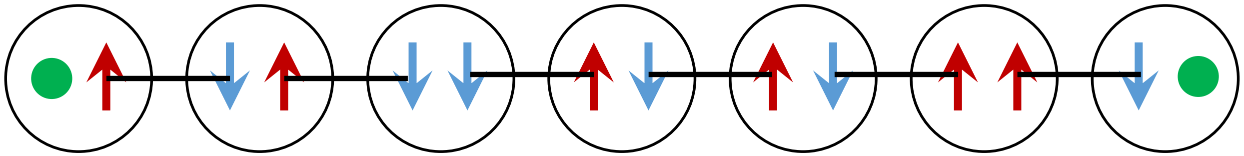
$Z_2 \times Z_2$ symmetry (π rotation about x,y,z axes)

Hidden $Z_2 \times Z_2$ symmetry breaking

With this transformation,

String order in $\mathcal{H}$
$\Leftrightarrow$ Ferromagnetic order in $\tilde{\mathcal{H}}$

4-fold degeneracy in $\mathcal{H}$
$\Leftrightarrow$ Edge state in $\tilde{\mathcal{H}}$



For general S, edge spin degeneracy is $(S+1)^2$.
In S=even case, Hidden $Z_2 \times Z_2$ symmetry breaking seems incompatible.

Is the string order enough?

Z.-C. Gu and X. G. Wen, PRB **80**, 155131 (2009)

No.

The Haldane phase is more "robust" than $Z_2 \times Z_2$.

$$\mathcal{H} = \sum_j J \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \sum_j D (S_j^z)^2 - \sum_j B S_j^x$$

String order cannot be defined.

Still, Haldane and large-D are "different" phases.

Symmetry protection of S=1 AF chain

F. Pollmann et al., PRB **81**, 064439 (2010); PRB **85**, 075125 (2012).

One of the following can protect the Haldane phase.

A) Dihedral ($Z_2 \times Z_2$) symmetry

B) Time-reversal symmetry

C) Bond-centered inversion symmetry

Matrix product state (MPS) representation is useful for the discussion.

Matrix product state

$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_N} A[i_1]A[i_2]\cdots A[i_N]|i_1, i_2, \ldots, i_N\rangle$$

↖ matrices

$|i\rangle \quad (i = 1, \ldots, d)$: d.o.f. on each site, e.g. $i = \uparrow, \downarrow$

Ex1: $|\psi\rangle = |000\ldots0\rangle + |111\ldots1\rangle$

$$A[0] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A[1] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex2: $|\psi\rangle = |100\ldots0\rangle + |010\ldots0\rangle \cdots + |000\ldots1\rangle$

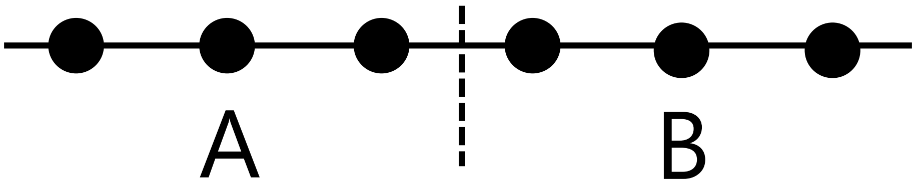
$$A[0] = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & \mathbf{0} & \\ & & 0 & \ddots & & \\ & & & \ddots & \ddots & \\ \mathbf{0} & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \quad A[1] = \begin{pmatrix} & & \\ & \mathbf{0} & \\ 1 & & \end{pmatrix}$$

Construction of MPS

A general way to obtain MPS of some state $|\psi\rangle$

Schmidt decomposition

$$|\psi\rangle = \sum_{i,j} M_{i,j} |i\rangle_A \, |j\rangle_B$$



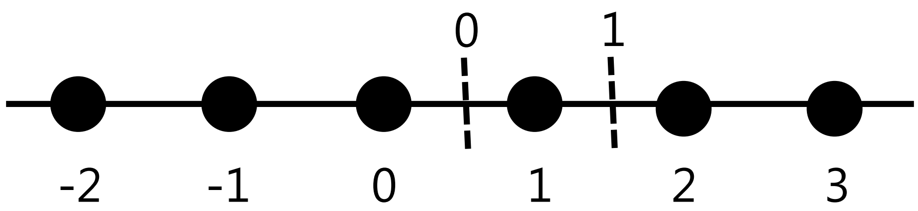
$$M_{i,j} = \sum_{\alpha} U_{i,\alpha} \Lambda_\alpha V^\dagger_{\alpha,j}$$: singular value decomposition

Λ : diagonal U, V : unitary

$$|\phi\rangle_A = U^\dagger |i\rangle_A, \; |\omega\rangle_B = V^\dagger |j\rangle_B$$

$$|\psi\rangle = \sum_{\alpha} \Lambda_\alpha |\phi\rangle_A \, |\omega\rangle_B$$

Construction of MPS

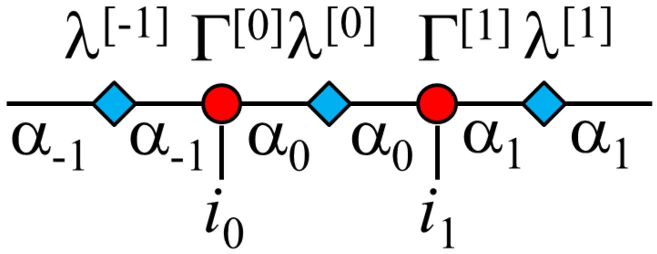


Schmidt decomp.

$$|\psi\rangle = \sum_{\alpha_0} \lambda^{[0]}_{\alpha_0} |\phi^{[\triangleleft 0]}_{\alpha_0}\rangle |\phi^{[1 \triangleright]}_{\alpha_0}\rangle \qquad |\psi\rangle = \sum_{\alpha_1} \lambda^{[1]}_{\alpha_1} |\phi^{[\triangleleft 1]}_{\alpha_1}\rangle |\phi^{[2 \triangleright]}_{\alpha_1}\rangle$$

$\Gamma^{[1]}_{\alpha_0 i_1 \alpha_1}$ is defined as $|\phi^{[\triangleleft 1]}_{\alpha_1}\rangle = \sum_{i_1, \alpha_0} \Gamma^{[1]}_{\alpha_0 i_1 \alpha_1} |\phi^{[\triangleleft 0]}_{\alpha_0}\rangle \lambda^{[0]}_{\alpha_0} |i_1\rangle$

Diagrammatic representation



solid line = summation

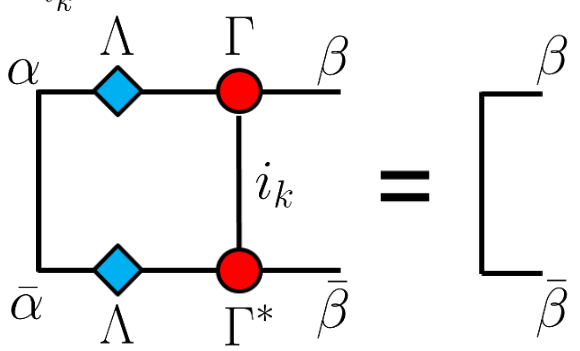
Canonical form

$$|\Psi\rangle = \sum_{\{i_k\}} \ldots \Lambda\Gamma[i_{k-1}]\Lambda\Gamma[i_k]\Lambda\Gamma[i_{k+1}]\Lambda \ldots | \ldots, i_{k-1}, i_k, i_{k+1}, \ldots\rangle$$

(Left) transfer matrix $T_{(\alpha,\bar{\alpha}),(\beta,\bar{\beta})} = \sum_{i_k} \Lambda_\alpha \Gamma_{\alpha,\beta}[i_k] \Lambda_{\bar{\alpha}} \Gamma^*_{\bar{\alpha},\bar{\beta}}[i_k]$

Canonical condition

$$\sum_{\alpha,\bar{\alpha}} \delta_{\alpha,\bar{\alpha}} T_{(\alpha,\bar{\alpha}),(\beta,\bar{\beta})} = \delta_{\beta,\bar{\beta}}$$



1 is the largest norm and nondegenerate eigenvalue of T

Degrees of freedom of MPS

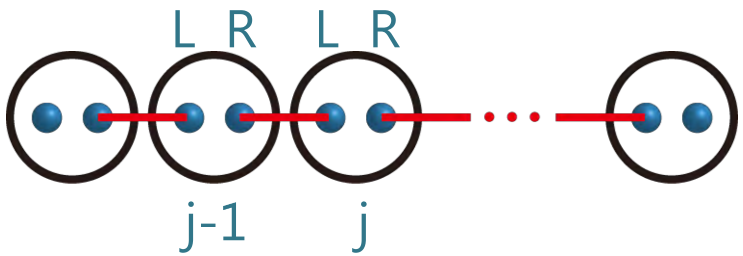
D. Pérez-García, et al., PRL **100**, 167202 (2008)

Phase factor: $e^{i\theta}$

Unitary transformation: $\Gamma \rightarrow U^\dagger \Gamma U \; (\Lambda U = U\Lambda)$

MPS for AKLT state

S=1



(j-1,R)-(j,L) Spin-1 Proj (j,R)-(j+1,L)

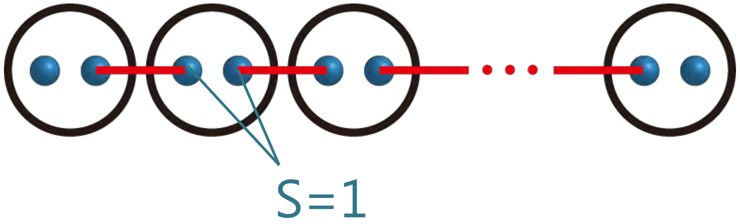
$$|\Psi\rangle = \cdots \begin{pmatrix} 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} |+1\rangle_j & |0\rangle_j/\sqrt{2} \\ |0\rangle_j/\sqrt{2} & |-1\rangle_j \end{pmatrix} \begin{pmatrix} 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{pmatrix} \cdots$$

$$\begin{pmatrix} |0\rangle_j/2 & |-1\rangle_j/\sqrt{2} \\ -|+1\rangle_j/\sqrt{2} & -|0\rangle_j/2 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \quad \Gamma[1] = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad \Gamma[0] = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{pmatrix} \quad \Gamma[-1] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

MPS for AKLT state

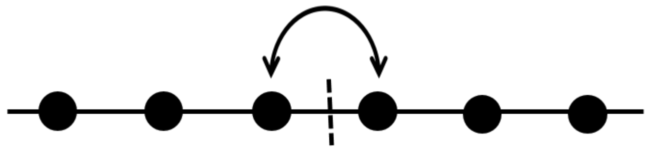
S=2



$$\Lambda = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{pmatrix} \quad \Gamma[2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{6} & 0 & 0 \end{pmatrix} \quad \Gamma[1] = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$\Gamma[0] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Gamma[-1] = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \quad \Gamma[-2] = \begin{pmatrix} 0 & 0 & \sqrt{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

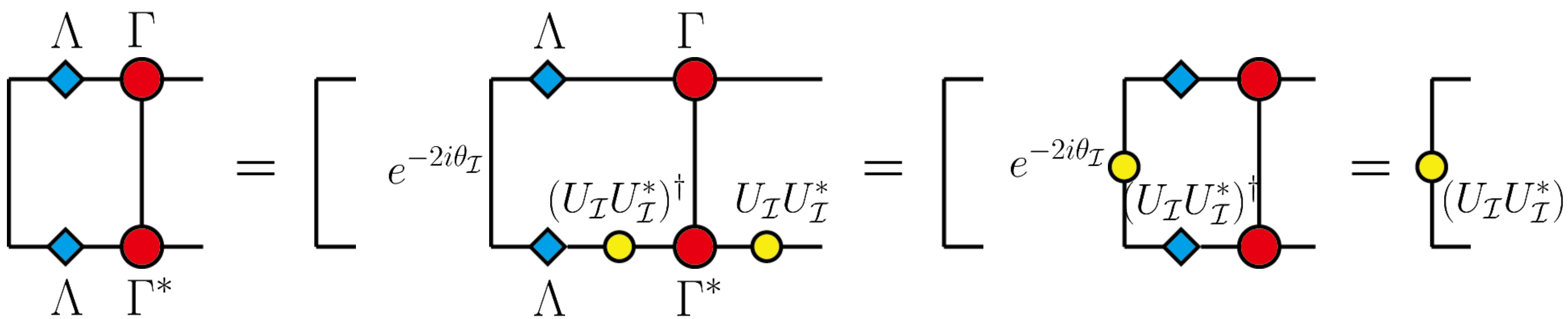
Inversion symmetry



$$(A_1 A_2 \cdots A_n)^{\mathrm{T}} = A_n^{\mathrm{T}} \cdots A_2^{\mathrm{T}} A_1^{\mathrm{T}}$$

Inversion $\mathcal{I}$ acts on MPS as $\Gamma^{\mathrm{T}} = e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^{\dagger} \Gamma U_{\mathcal{I}}$

$$\left.\begin{aligned} \Gamma^{\mathrm{T}} &= e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^{\dagger} \Gamma U_{\mathcal{I}} \\ \Gamma &= e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^{\mathrm{T}} \Gamma^{\mathrm{T}} U_{\mathcal{I}}^{*} \end{aligned}\right\} \quad \begin{aligned} \Gamma &= e^{2i\theta_{\mathcal{I}}} (U_{\mathcal{I}}^{*} U_{\mathcal{I}})^{\dagger} \Gamma U_{\mathcal{I}}^{*} U_{\mathcal{I}} \\ \Gamma^{*} &= e^{-2i\theta_{\mathcal{I}}} (U_{\mathcal{I}} U_{\mathcal{I}}^{*})^{\dagger} \Gamma^{*} U_{\mathcal{I}} U_{\mathcal{I}}^{*} \end{aligned}$$



$$e^{2i\theta_{\mathcal{I}}} = 1$$

$$(U_{\mathcal{I}} U_{\mathcal{I}}^{*})^{\dagger} = e^{i\phi} E \rightarrow U_{\mathcal{I}}^{\mathrm{T}} = e^{i\phi} U_{\mathcal{I}} \rightarrow U_{\mathcal{I}}^{\mathrm{T}} = \pm U_{\mathcal{I}}$$

Inversion symmetry

S=1

$$\Lambda = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \quad \Gamma[1] = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad \Gamma[0] = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{pmatrix} \quad \Gamma[-1] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\Gamma^{\mathrm{T}} = e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^{\dagger} \Gamma U_{\mathcal{I}}$$

You can find $U_{\mathcal{I}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\Gamma^{\mathrm{T}} = -U_{\mathcal{I}}^{\dagger} \Gamma U_{\mathcal{I}}$$

$U_{\mathcal{I}} = -U_{\mathcal{I}}^{\mathrm{T}}$: Nontrivial

Inversion symmetry

S=2

$$\Lambda = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{pmatrix} \quad \Gamma[2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{6} & 0 & 0 \end{pmatrix} \quad \Gamma[1] = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$\Gamma[0] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Gamma[-1] = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \quad \Gamma[-2] = \begin{pmatrix} 0 & 0 & \sqrt{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{\mathrm{T}} = e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^{\dagger} \Gamma U_{\mathcal{I}}$$

You can find $U_{\mathcal{I}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\Gamma^{\mathrm{T}} = +U_{\mathcal{I}}^{\dagger} \Gamma U_{\mathcal{I}}$$

$U_{\mathcal{I}} = +U_{\mathcal{I}}^{\mathrm{T}}$: Trivial

Time-reversal symmetry

$$|\Psi\rangle = \sum_{S_j^z} \ldots \Lambda\Gamma[S_{j-1}^z]\Lambda\Gamma[S_j^z]\Lambda\Gamma[S_{j+1}^z]\Lambda \ldots \bigotimes_j |S_j^z\rangle$$

Time-reversal operation $e^{i\pi S_y}K$

Complex conjugation

$$e^{i\pi S_y}\Gamma^* = e^{i\theta_\mathcal{T}} U_\mathcal{T}^\dagger \Gamma U_\mathcal{T}$$

$$\left.\begin{aligned} \Gamma^* &= e^{i\theta_\mathcal{T}} U_\mathcal{T}^\dagger e^{i\pi S_y} \Gamma U_\mathcal{T} \\ \Gamma &= e^{-i\theta_\mathcal{T}} U_\mathcal{T}^\mathrm{T} e^{i\pi S_y} \Gamma^* U_\mathcal{T}^* \end{aligned}\right\} \begin{aligned} \Gamma &= e^{i2\pi S_y} (U_\mathcal{T} U_\mathcal{T}^*)^\dagger \Gamma U_\mathcal{T} U_\mathcal{T}^* \\ &= (U_\mathcal{T} U_\mathcal{T}^*)^\dagger \Gamma U_\mathcal{T} U_\mathcal{T}^* \end{aligned}$$

$$(U_\mathcal{T}^* U_\mathcal{T})^\dagger = e^{i\phi} E \rightarrow U_\mathcal{T}^\mathrm{T} = e^{i\phi} U_\mathcal{T} \rightarrow U_\mathcal{T}^\mathrm{T} = \pm U_\mathcal{T}$$

Same as inversion

$Z_2 \times Z_2$ symmetry

π-rotation about spin x,y,z-axis forms $Z_2 \times Z_2$ group

$$\{1, R_x^\pi, R_y^\pi, R_z^\pi\} \;\; (R_z^\pi = R_x^\pi R_y^\pi)$$

$$R_x^\pi \Gamma = e^{i\theta_x} U_x^\dagger \Gamma U_x$$

$$\Gamma = (R_x^\pi)^2 \Gamma = e^{2i\theta_x} (U_x^2)^\dagger \Gamma U_x^2$$

$$e^{2i\theta_x} = 1 \qquad U_x^2 = e^{i\phi} E$$

Only one π-rotation does not protect the phase.

$$R_x^\pi R_y^\pi = R_y^\pi R_x^\pi$$

$$\left.\begin{aligned} R_x^\pi R_y^\pi \Gamma &= e^{i(\theta_x+\theta_y)} (U_x U_y)^\dagger \Gamma U_x U_y \\ R_y^\pi R_x^\pi \Gamma &= e^{i(\theta_x+\theta_y)} (U_y U_x)^\dagger \Gamma U_y U_x \end{aligned}\right\} \; \Gamma = (U_x U_y U_x^\dagger U_y^\dagger)^\dagger \Gamma U_x U_y U_x^\dagger U_y^\dagger$$

$$U_x U_y U_x^\dagger U_y^\dagger = e^{i\phi_{xy}} E \to U_x U_y = e^{i\phi_{xy}} U_y U_x$$

$$U_x^2 U_y = e^{i\phi_{xy}} U_x U_y U_x = e^{i2\phi_{xy}} U_y U_x^2 \qquad U_x U_y = \pm U_y U_x$$

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- Strange correlator

  Indicator for SPT states

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Nonlinear sigma model

(1+1) D Heisenberg antiferromagnet (Spin-S)

$$\mathcal{H} = \sum_j J \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \ (J > 0)$$

Effective field theory — O(3) nonlinear sigma model

$$\boldsymbol{S}_j / S \sim (-1)^j \boldsymbol{n}(x) + (a/S)\boldsymbol{l}(x)$$

$$\mathcal{S}[\boldsymbol{n}] = \frac{1}{2g}\int d\tau dx \left\{ \frac{1}{v}(\partial_\tau \boldsymbol{n})^2 + v(\partial_x \boldsymbol{n})^2 \right\} + i\theta Q_{\tau x} \quad |\boldsymbol{n}| = 1$$

$$Q_{\tau x} = \frac{1}{4\pi}\int d\tau dx \boldsymbol{n} \cdot \partial_\tau \boldsymbol{n} \times \partial_x \boldsymbol{n} \in \mathbb{Z} \qquad g = 2/S \quad v = 2JS$$

Haldane's argument F. D. M. Haldane (2008)

$\theta = 2\pi S \equiv 0 \pmod{2\pi}$ Integer spin (gapped)

$\theta \equiv \pi \pmod{2\pi}$ Half-odd integer spin (gapless, critical)

Ground state wave functional

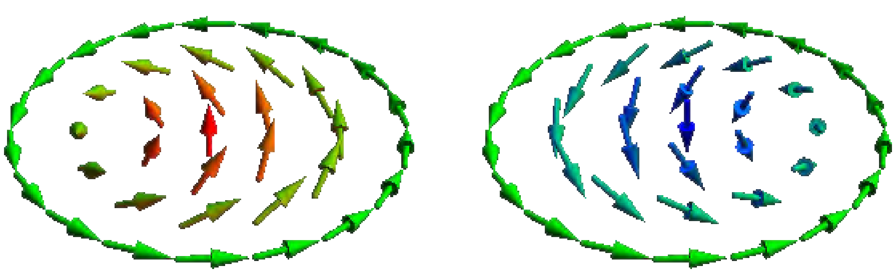
What is the difference between
S=odd and even?
-> See the ground state wave functional.

Easy plane AF $\mathcal{H} = \sum_j J\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \sum_j D(S_j^z)^2 \ (D > 0)$

$$\mathcal{S}[\boldsymbol{n}] = \frac{1}{2g}\int d\tau dx \left\{\frac{1}{v}(\partial_\tau \boldsymbol{n})^2 + v(\partial_x \boldsymbol{n})^2\right\} + i\theta Q_{\tau x}$$

$$\theta = 2\pi S$$

Meron configuration $Q_{\tau x} = \pm 1/2$



Ground state wave functional

$$\mathcal{S}[\boldsymbol{n}] = \frac{1}{2g}\int d\tau dx \left\{\frac{1}{v}(\partial_\tau \boldsymbol{n})^2 + v(\partial_x \boldsymbol{n})^2\right\} + i\theta Q_{\tau x}$$

$$Q_{\tau x} = \pm 1/2 \qquad \theta = 2\pi S$$

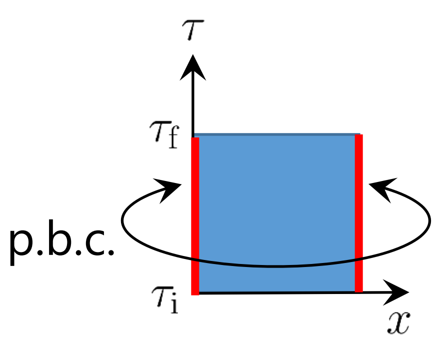
Strong coupling limit $g \to \infty$

$$|\Psi\rangle = \sum_{\{\boldsymbol{n}(x)\}} \Psi[\boldsymbol{n}(x)]|\boldsymbol{n}(x)\rangle$$

Path integral formalism

$$\Psi[\boldsymbol{n}(x)] \propto \int_{\boldsymbol{n}_\mathrm{i}}^{\boldsymbol{n}_\mathrm{f}=\boldsymbol{n}(x)} \mathcal{D}\boldsymbol{n}'(\tau,x) e^{-\mathcal{S}[\boldsymbol{n}'(\tau,x)]}$$

$$\sim \int_{\boldsymbol{n}_\mathrm{i}}^{\boldsymbol{n}(x)} \mathcal{D}\boldsymbol{n}'(\tau,x) e^{-i\theta \sum Q_{\tau x}}$$

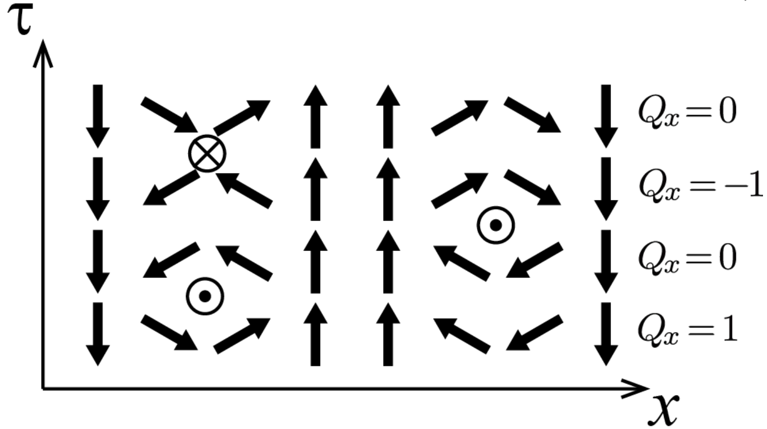


Ground state wave functional

$$\Psi[\boldsymbol{n}(x)] \propto \int_{\boldsymbol{n}_\mathrm{i}}^{\boldsymbol{n}(x)} \mathcal{D}\boldsymbol{n}'(\tau, x) e^{-i2\pi S \sum Q_{\tau x}} \quad Q_{\tau x} = \pm 1/2$$

S=even $\quad e^{-i2\pi S \sum Q_{\tau x}} = 1$

S=odd $\quad e^{-i2\pi S \sum Q_{\tau x}} = (-1)^{\sum 2Q_{\tau x}}$



$$Q_x \equiv \frac{1}{2\pi} \int dx \partial_x \phi(x) \in \mathbb{Z}$$

Winding number of the planar config.

$$\boxed{\Psi[\phi(x)] \propto e^{-iS\pi Q_x}} = \begin{cases} (-1)^{Q_x} & \text{if } S = \text{odd} \\ 1 & \text{if } S = \text{even} \end{cases}$$

Dual vortex theory

Useful for the discussion of protecting symmetry

$$\mathcal{S} = \frac{1}{2g}\int d\tau dx (\partial_\mu \phi)^2 + i2\pi S Q_{\tau x} \qquad Q_{\tau x} = \frac{1}{2} q Q_{\mathrm{vor}}$$

$q = \pm 1$: Core $\qquad Q_{\mathrm{vor}} \in \mathbb{Z}$: vorticity

Hubbard-Stratonovich transformation

$$\frac{1}{2g}(\partial_\mu \phi)^2 \to \frac{g}{2} J_\mu^2 + i J_\mu \partial_\mu \phi$$

$\phi = \phi_{\mathrm{r}} + \phi_{\mathrm{v}} \quad (\partial_\tau \partial_x - \partial_x \partial_\tau)\phi_{\mathrm{r}} = 0$: regular part

$(\partial_\tau \partial_x - \partial_x \partial_\tau)\phi_{\mathrm{v}} \neq 0$: vortex part

Integration over ϕ_{r} $\longrightarrow$ Delta function $\propto \delta(\partial_\mu J_\mu)$

$J_\mu = \epsilon_{\mu\nu} \partial_\nu \varphi / 2\pi \qquad \varphi$: vortex free scalar field

$$i J_\mu \partial_\mu \phi \to -i \varphi \rho_{\mathrm{vor}} \qquad \rho_{\mathrm{vor}} = \epsilon_{\mu\nu} \partial_\mu \partial_\nu / 2\pi$$

Dual vortex theory

$$\mathcal{S} = \frac{g}{8\pi^2}\int d\tau dx (\partial_\mu \varphi)^2 + i(q\pi S - \varphi) Q_{\rm vor}$$

Small fugacity expansion $z = e^{-\mu}$

μ : creation energy of a vortex

Dual action

$$\mathcal{S}_{\rm dual}[\varphi(\tau,x)] = \int d\tau dx \left[\frac{g}{8\pi^2}(\partial_\mu\varphi)^2 - 4z\cos(\pi S)\cos\varphi\right]$$

For integer-S,

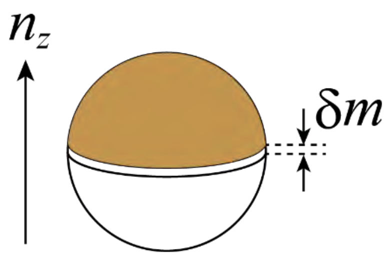
$$\mathcal{S}_{\rm dual}[\varphi(\tau,x)] = \int d\tau dx \left[\frac{g}{8\pi^2}(\partial_\mu\varphi)^2 - 4z(-1)^S\cos\varphi\right]$$

sine-Gordon model

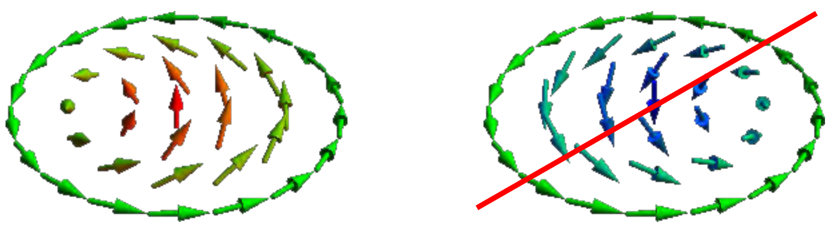
SPT breaking perturbation

Staggered field changes z-component by δm

Meron contribution is shifted $i\pi S \to i\pi(S-\delta m)$



In addition, the meron core is fixed $q=1$



Dual theory is modified as

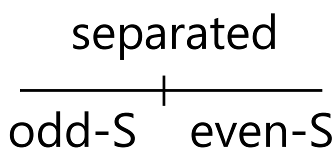
$$\mathcal{S}_{\mathrm{dual}}[\varphi(\tau,x)] = \int d\tau dx \left[\frac{g}{8\pi^2}(\partial_\mu\varphi)^2 - 2z\cos(\pi(S-\delta m)-\varphi)\right]$$

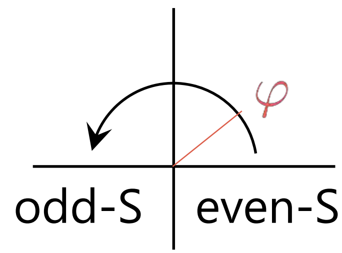
SPT breaking perturbation

$$\mathcal{S}_{\rm dual}[\varphi(\tau,x)] = \int d\tau dx \left[\frac{g}{8\pi^2}(\partial_\mu\varphi)^2 - 4z(-1)^S\cos\varphi\right]$$

$$\Rightarrow \mathcal{S}_{\rm dual}[\varphi(\tau,x)] = \int d\tau dx \left[\frac{g}{8\pi^2}(\partial_\mu\varphi)^2 - 2z\cos(\pi(S-\delta m)-\varphi)\right]$$

For z>0





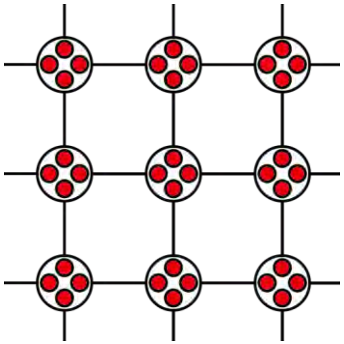
Phase is locked at $\varphi = \pi(S-\delta m)$

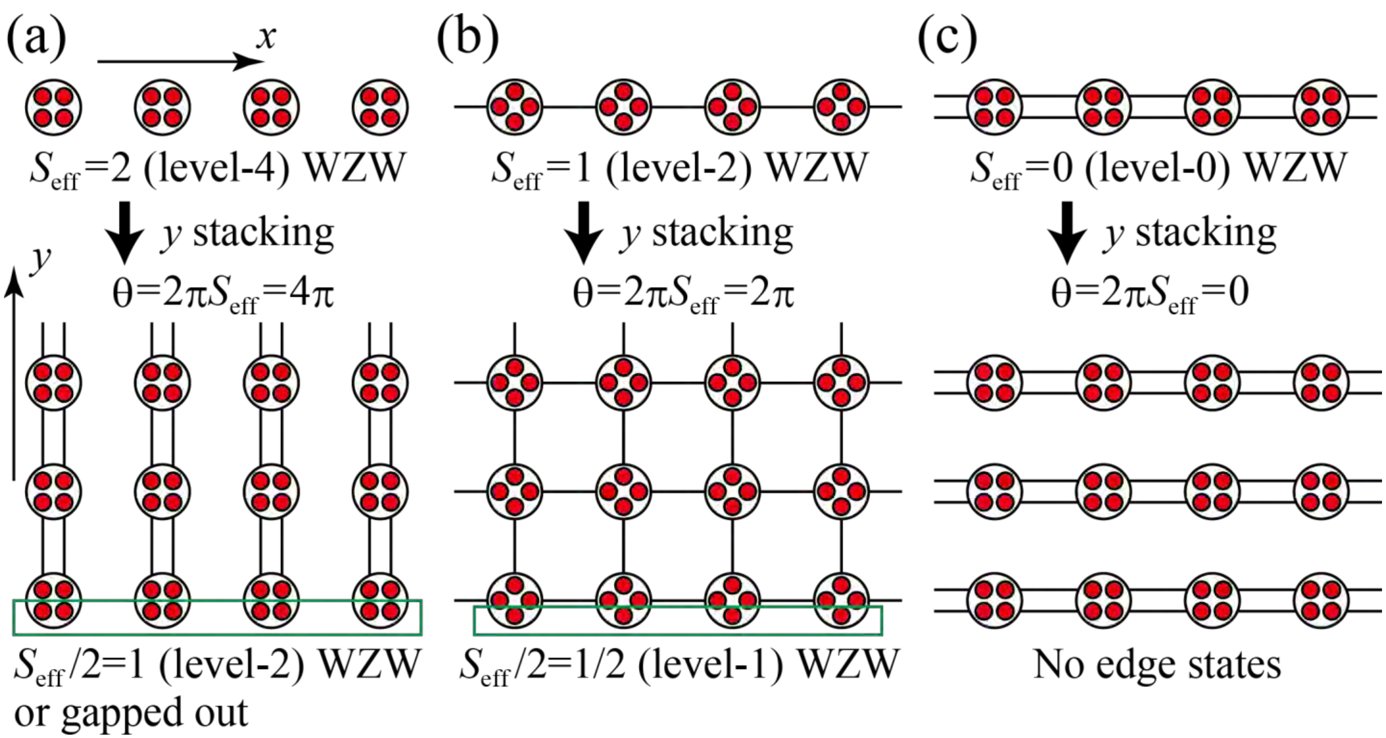
S = even and odd are continuously connected by changing δm. Staggered field breaks

A) Dihedral ($Z_2 \times Z_2$) symmetry
B) Time-reversal symmetry
C) Bond-centered inversion symmetry

2D AKLT state

ST, P. Pujol, and A. Tanaka,
Phys. Rev. B **94**, 235159 (2016).





2D AKLT state

1D-2D analogy

	(1+1)D easy plane Haldane state	(2+1)D VBS states
target manifold	S^1 (planar)	S^2 (spherical)
topological term at spatial edge	theta term of (0+1)D O(2) NLσM: $\mathcal{S}_\Theta^{\text{edge}} = i\pi S Q_\tau$, $Q_\tau \equiv \frac{1}{2\pi}\int d\tau \partial_\tau \phi$	theta term of (1+1)D O(3) NLσM: $\mathcal{S}_\Theta^{y\text{-edge}} = i\pi (S/2) Q_{\tau x}$, etc., $Q_{\tau x} \equiv \frac{1}{4\pi}\int d\tau dx \boldsymbol{n}\cdot\partial_\tau \boldsymbol{n}\times\partial_x \boldsymbol{n}$
winding # (snapshot config.)	$Q_x \equiv \frac{1}{2\pi}\int dx \partial_x \phi$	$Q_{xy} \equiv \frac{1}{4\pi}\int dx dy \boldsymbol{n}\cdot\partial_x \boldsymbol{n}\times\partial_y \boldsymbol{n}$
singular space-time event	vortex (phase-slip) $\Delta_\tau Q_x \neq 0$	monopole $\Delta_\tau Q_{xy} \neq 0$
vacuum wave functional	$\Psi[\phi(x)] \propto e^{-i\pi S Q_x}$	$\Psi[\boldsymbol{n}(x,y)] \propto e^{-i\pi \frac{S}{2} Q_{xy}}$
Distinction	S = even vs odd	$S = 2,6,\ldots$ vs $4,8,\ldots$

Outline

- Introduction
  - What is SPT?
- SPT state in 1D antiferromagnets
  - AKLT VBS state, Haldane phase, MPS
- Field theory of SPT state
  - Nonlinear sigma model, GS wave functional
- Strange correlator
  - Indicator for SPT states
- Conclusion

Strange Correlator

- Definition

$$C_S(\boldsymbol{R},0) \equiv \frac{\langle\Psi_0|\hat{\boldsymbol{n}}(\boldsymbol{R})\cdot\hat{\boldsymbol{n}}(0)|\Psi\rangle}{\langle\Psi_0|\Psi\rangle}$$

$|\Psi\rangle$: Ground state

$|\Psi_0\rangle$: Trivial (direct product) state

At $|\boldsymbol{R}| \to \infty$ $\begin{cases} \text{nonzero or power-law decay: SPT} \\ \text{Exponential decay: Trivial} \end{cases}$

- Idea

e.g. 2d case $\Psi[\boldsymbol{n}(\boldsymbol{r})] = Ne^{-W[\boldsymbol{n}(\boldsymbol{r})]}$

$$W[\boldsymbol{n}(\boldsymbol{r})] \equiv \int d^2\boldsymbol{r}\left[\frac{1}{2\tilde{g}}(\partial_\alpha\boldsymbol{n})^2 + i\frac{\Theta}{4\pi}\boldsymbol{n}\cdot\partial_x\boldsymbol{n}\times\partial_y\boldsymbol{n}\right]$$

Usual two-point correlator

$$C(\boldsymbol{R},0) = \langle\Psi|\hat{\boldsymbol{n}}(\boldsymbol{R})\cdot\hat{\boldsymbol{n}}(0)|\Psi\rangle = \int\mathcal{D}\boldsymbol{n}(\boldsymbol{r})|\Psi[\boldsymbol{n}(\boldsymbol{r})]|^2\boldsymbol{n}(\boldsymbol{R})\cdot\boldsymbol{n}(0)$$

No topological effect

Strange correlator

$$C_S(\boldsymbol{R},0) \equiv \frac{\langle\Psi_0|\hat{\boldsymbol{n}}(\boldsymbol{R})\cdot\hat{\boldsymbol{n}}(0)|\Psi\rangle}{\langle\Psi_0|\Psi\rangle}$$

Effects from the theta term

1d case

Strange correlator

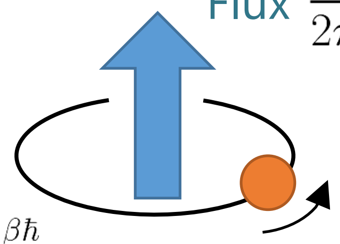
$$C_{\mathrm{S}}(X,0) \equiv \frac{1}{Z}\int_{\mathrm{pbc}} \mathcal{D}\phi(x)\cos\phi(X)\cos\phi(0)e^{-\frac{1}{\hbar}\int dx[\frac{\hbar^2}{\tilde{g}}(\partial_x\phi)^2+i\hbar\frac{\Theta}{2\pi}\partial_x\phi]}$$

$$Z \equiv \int_{\mathrm{pbc}} \mathcal{D}\phi(x)e^{-\frac{1}{\hbar}\int dx[\frac{\hbar^2}{\tilde{g}}(\partial_x\phi)^2+\underline{i\hbar\frac{\Theta}{2\pi}\partial_x\phi}]}$$

Aharonov-Bohm phase

Relabeling of coordinate $x \to \tau$

Calculation of imaginary time correlator of a particle on a ring with flux



Flux $\frac{\Theta}{2\pi}$

$$Z = \sum_{m\in\mathbb{Z}} e^{-im\Theta}\int_m \mathcal{D}\phi(\tau)e^{-\int d\tau\frac{\hbar}{\tilde{g}}(\partial_\tau\phi)^2} \qquad m \equiv \frac{1}{2\pi}\int_0^{\beta\hbar}\partial_\tau\phi \in \mathbb{Z}$$

$$\hat{\mathcal{H}} = \frac{\tilde{g}}{4\hbar^2}\left(\hat{\pi}+\frac{\hbar\Theta}{2\pi}\right)^2 = \frac{\tilde{g}}{4}\left(\hat{N}-\frac{\Theta}{2\pi}\right)^2 \qquad \hat{\pi} = -i\hbar\partial_\phi$$

$$\hat{N}\psi_n = n\psi_n \qquad \psi_n(\phi) = \langle\phi|n\rangle = \frac{1}{\sqrt{2\pi}}e^{-in\phi}$$

$$E_n = \frac{\tilde{g}}{4}(n-\frac{\Theta}{2\pi})^2$$

1d case

$$\psi_n(\phi) = \langle\phi|n\rangle = \frac{1}{\sqrt{2\pi}}e^{-in\phi} \qquad E_n = \frac{\tilde{g}}{4}(n - \frac{\Theta}{2\pi})^2 \qquad \Theta = \pi S$$

$$C_{\mathrm{S}}(\tau, 0) \equiv \langle\cos\hat{\phi}(\tau)\cos\hat{\phi}(0)\rangle$$

$$\langle\hat{O}(\tau)\hat{O}(0)\rangle = \langle G|e^{\frac{\tau}{\hbar}\hat{H}}\hat{O}e^{-\frac{\tau}{\hbar}\hat{H}}\hat{O}|G\rangle = \sum_n e^{-\frac{\tau}{\hbar}(E_n - E_0)}|\langle n|\hat{O}|G\rangle|^2$$

$$\langle n|e^{\pm i\hat{\phi}}|0\rangle = \delta_{n,\pm 1}$$

(i) $\Theta = 0$ case

$$|G\rangle = |0\rangle$$

$$C_{\mathrm{S}}(\tau, 0) = \frac{1}{2}e^{-\frac{\tilde{g}\tau}{4\hbar}}$$ exp. decay: trivial phase

(ii) $\Theta = \pi$ case

$$|G\rangle = c_0|0\rangle + c_1|1\rangle$$

$$C_{\mathrm{S}}(\tau, 0) = \frac{1}{4}(1 + e^{-\frac{\tilde{g}\tau}{2\hbar}})$$ Nonzero at $\tau \to \infty$: SPT phase

1d case (Remark)

Strange correlator of 1d AKLT state can be calculated exactly using MPS

$$|\Psi\rangle = \sum_{\{\sigma_n\}} A[\sigma_1]A[\sigma_2]\cdots A[\sigma_N]|\{\sigma_n\}\rangle \qquad |\Psi_0\rangle = |0\,0\,0\ldots\rangle$$

S=1

$$A[1] = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, A[0] = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, A[-1] = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$\langle\Psi_0|\Psi\rangle = \mathrm{Tr}(A[0]^N) = 2^{-(N-1)}$$

$$\begin{aligned}\langle\Psi_0|S_i^+S_j^-|\Psi\rangle &= 2\mathrm{Tr}(A[0]^{i-1}A[-1]A[0]^{j-i-1}A[1]A[0]^{N-j}) \\ &= (-1)^{j-i}2^{2-N}\end{aligned}$$

$C_\mathrm{S}(i,j) = 2(-1)^{j-i}$ Nonzero at $|j-i| \to \infty$

1d case (Remark)

S=2

$$A[1] = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \end{pmatrix}, A[0] = \frac{1}{3\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A[-1] = \begin{pmatrix} 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 \end{pmatrix}$$

$C_{\mathrm{S}}(i,j) \sim 9(-1/2)^{i-j}$ exp. decay

S=3

$C_{\mathrm{S}}(i,j) \sim 8(-1)^{j-i}$ Nonzero at $|j-i| \to \infty$

2d case

$$C_{\mathrm{S}}(\boldsymbol{R}) = \frac{\displaystyle\int \mathcal{D}\boldsymbol{n}(\boldsymbol{r}) e^{-W[\boldsymbol{n}(\boldsymbol{r})]} \boldsymbol{n}(\boldsymbol{R}) \cdot \boldsymbol{n}(0)}{\displaystyle\int \mathcal{D}\boldsymbol{n}(\boldsymbol{r}) e^{-W[\boldsymbol{n}(\boldsymbol{r})]}}$$

$$W[\boldsymbol{n}(\boldsymbol{r})] \equiv \int d^2\boldsymbol{r} \left[\frac{1}{2\tilde{g}}(\partial_\alpha \boldsymbol{n})^2 + i\frac{\Theta}{4\pi}\boldsymbol{n}\cdot\partial_x\boldsymbol{n}\times\partial_y\boldsymbol{n}\right] \qquad \Theta = \frac{\pi S}{2}$$

Relabeling of coordinate $y \to \tau$

Strange correlator corresponds to two point correlator
in (1+1)d nonlinear sigma model + theta term

S=2,6,... $\Theta \equiv \pi \pmod{2\pi}$ half-odd integer spin chain (gapless)
power-law decay

S=4,8,... $\Theta \equiv 0 \pmod{2\pi}$ integer spin chain (gapped)
exp. decay

Strange correlator correctly distinguishes SPT state

Conclusion

- SPT phase is protected only if some symmetry is imposed on the system. (No LRE, No SSB)
- Typical example: S=1 AF chain. To discuss the SPT phase, MPS is useful. String order for the $Z_2 \times Z_2$ case.
- Field theory: NLSM+topo. term. SPT property appears in GS wave functional.
- Strange correlator: indicator of SPT.

$$C_{\mathrm{S}}(\boldsymbol{R},0) \equiv \frac{\langle \Psi_0 | \hat{\boldsymbol{n}}(\boldsymbol{R}) \cdot \hat{\boldsymbol{n}}(0) | \Psi \rangle}{\langle \Psi_0 | \Psi \rangle}$$