Optimal transport: From stochastic thermodynamics to quantum many-body systems
最適輸送：ゆらぎ熱力学から量子多体系まで

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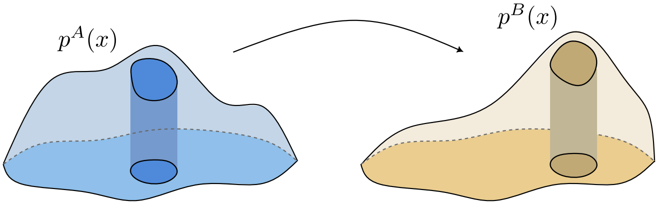
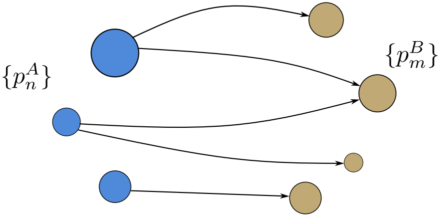
Optimal transport

About the optimal planning and optimal cost of transporting distributions





Monge formulation of optimal transport

Monge formulation

Optimal transport cost with respect to a cost function $c(x,y) : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}_{\geq 0}$:

$$M(p^A, p^B) \coloneqq \min_{\varphi} \int c(x, \varphi(x)) p^A(x)\, dx$$

$\varphi : \mathbb{R}^d \mapsto \mathbb{R}^d$: one-to-one map satisfying $p^A(x) = p^B(\varphi(x))|\nabla\varphi(x)|$

φ^*: optimal transport map

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- Non-existence of a valid transport map: φ^* might not exist in discrete cases because no mass can be split

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- Resolved by the relaxation of Kantorovich

Kantorovich formulation of optimal transport

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$\pi : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}_{>0}$: coupling of p^A and p^B (a joint probability distribution function of x and y)

$$\int_{\mathbb{R}^d} \pi(x, y)\, dy = p^A(x) \text{ and } \int_{\mathbb{R}^d} \pi(x, y)\, dx = p^B(y)$$

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- Two formulations are equivalent when distributions are absolutely continuous

Wasserstein distances

L^α-Wasserstein distance

Optimal transport cost with respect to a cost function $c(x,y) = \|x-y\|^\alpha$:

$$W_\alpha(p^A, p^B)^\alpha \coloneqq \min_\pi \int \|x-y\|^\alpha \pi(x,y)\, dx\, dy$$

$\|\cdot\|$: Euclidean norm

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- W_1 and W_2 are of interest

Discrete Wasserstein distances

- Transport a N-dimensional distribution $p^A = [p^A_x]$ to distribution $p^B = [p^B_x]$ with respect to a cost matrix $C = [c_{xy}]$

Discrete Wasserstein distances

- Transport a N-dimensional distribution $p^A = [p_x^A]$ to distribution $p^B = [p_x^B]$ with respect to a cost matrix $C = [c_{xy}]$
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$$W_\alpha(p^A, p^B)^\alpha := \min_{\pi \in \Pi(p^A, p^B)} \sum_{x,y} c_{xy}^\alpha \pi_{xy}$$

  - $c_{xy} \geq 0$: cost of transporting a unit probability from p_y^A to p_x^B
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- Number of choices for the cost matrix C is infinite

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- Generalized Wasserstein distance [Piccoli et al., ARMA (2013)]

$$\mathcal{W}_{1,\lambda}(\boldsymbol{x}^a, \boldsymbol{x}^b) \coloneqq \min\left\{\lambda(\|\boldsymbol{x}^a - \tilde{\boldsymbol{x}}^a\|_1 + \|\boldsymbol{x}^b - \tilde{\boldsymbol{x}}^b\|_1) + \mathcal{W}_1(\tilde{\boldsymbol{x}}^a, \tilde{\boldsymbol{x}}^b)\right\}$$

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  - $W_{1,\lambda}$ satisfies the triangle inequality

Optimal transport and stochastic thermodynamics

Thermodynamics of continuous-variable optimal transport

Benamou–Brenier formula [Numer. Math. (2000)]

$$W_2(p^A, p^B)^2 = \min_{v_t} \tau \int_0^\tau \int_{\mathbb{R}^d} \|v_t(x)\|^2 p_t(x)\, dx\, dt$$

the minimum is over all smooth paths $\{v_t\}_{0 \le t \le \tau}$ subject to the continuity equation

$$\dot{p}_t(x) + \nabla \cdot [v_t(x) p_t(x)] = 0$$

with the initial and final conditions $p_0(x) = p^A(x)$ and $p_\tau(x) = p^B(x)$

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- For overdamped Fokker–Planck dynamics, $v_t(x) = F_t(x) - D\nabla \ln p_t(x)$ and

$$\Sigma_\tau = \frac{1}{D} \int_0^\tau \int_{\mathbb{R}^d} \|v_t(x)\|^2 p_t(x)\, dx\, dt$$

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- Wasserstein distance in terms of dissipation

$$W_2(p^A, p^B) = \min_{F_t} \sqrt{D\tau\Sigma_\tau}$$

Essential applications of Benamou–Brenier formula

- Mandelstam-Tamm (MT) and Margolus-Levitin (ML) speed limits inspired by Heisenberg uncertainty principle $\Delta t \times \Delta E \gtrsim \hbar$:

$$\tau \geq \frac{\pi}{2} \max \left\{ \frac{\hbar}{\Delta H}, \frac{\hbar}{\langle H \rangle - E_g} \right\}$$

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- Thermodynamic speed limit for overdamped Langevin dynamics [Aurell et al., JSP (2012)]

$$\tau \geq \frac{W_2(p_0, p_\tau)}{\sqrt{D \langle \sigma \rangle_\tau}}$$

$\langle \sigma \rangle_\tau := \tau^{-1} \Sigma_\tau$: time-average entropy production

Essential applications of Benamou–Brenier formula

Landauer principle

Minimum heat dissipation required for erasing of one bit of information

$$Q \geq k_B T \ln 2$$

T: the temperature of the environment

$$\cdots \mid 0 \mid 1 \mid 1 \mid 0 \mid 1 \mid 0 \mid \cdots$$

$$\Downarrow \Lambda(\text{input})$$

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- Finite-time Landauer principle

$$\beta Q \geq \ln 2 + \frac{W_2(p_0, p_\tau)^2}{D\tau}$$

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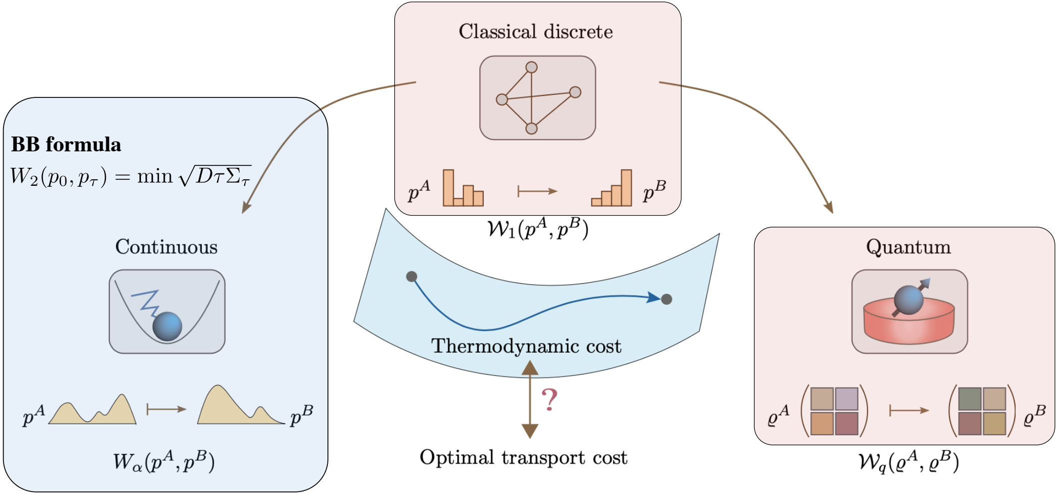
  - For 1D overdamped systems with double-well potentials [Proesman et al., PRL (2020)]

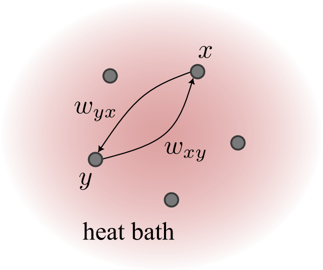
$$\beta Q \geq \ln 2 + \frac{\mathrm{Var}(x)}{2D\tau}$$

Motivation


BB formula
$W_2(p_0, p_\tau) = \min \sqrt{D\tau\Sigma_\tau}$
Continuous
p^A
p^B
$W_\alpha(p^A, p^B)$
Classical discrete
p^A
p^B
$\mathcal{W}_1(p^A, p^B)$
Thermodynamic cost
?
Optimal transport cost
Quantum
ϱ^A
ϱ^B
$\mathcal{W}_q(\varrho^A, \varrho^B)$

Markov jump processes

- Discrete-state system with N states: $\dot{p}_t = \mathsf{W}_t p_t$, $\mathsf{W}_t = [w_{xy}(t)]$



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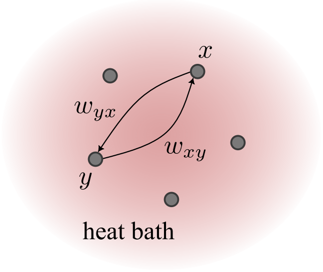
- Microscopically reversible: $w_{xy}(t) > 0$ whenever $w_{yx}(t) > 0$



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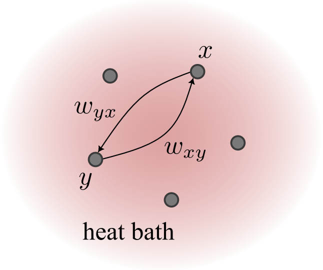
$$\ln \frac{w_{xy}(t)}{w_{yx}(t)} = s_{xy}(t)$$

$s_{xy}(t)$: environmental entropy change associated with jump $y \to x$



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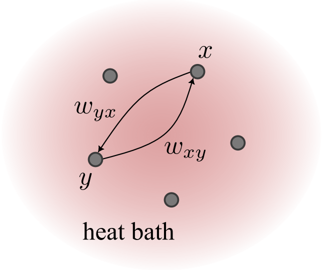

- Irreversible entropy production rate

$$\sigma_t := \dot{s}_{\mathrm{sys}}(t) + \dot{s}_{\mathrm{env}}(t) = \frac{1}{2}\sum_{x \neq y}[a_{xy}(t) - a_{yx}(t)] \ln \frac{a_{xy}(t)}{a_{yx}(t)} \geq 0$$

$$a_{xy}(t) := w_{xy}(t) p_y(t)$$

$$j_{xy}(t) := a_{xy}(t) - a_{yx}(t)$$

$$f_{xy}(t) := \ln \frac{a_{xy}(t)}{a_{yx}(t)}$$

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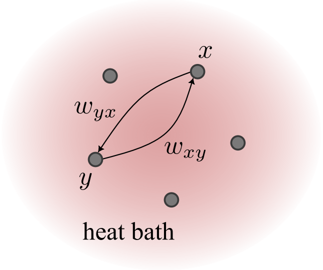

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- Dynamical activity $\mathcal{A}_\tau \coloneqq \int_0^\tau a_t\, dt$ quantifies the total number of jumps

$$a_t \coloneqq \sum_{x \neq y} a_{xy}(t)$$

$$a_{xy}(t) \coloneqq w_{xy}(t) p_y(t)$$

$$j_{xy}(t) \coloneqq a_{xy}(t) - a_{yx}(t)$$

$$f_{xy}(t) \coloneqq \ln \frac{a_{xy}(t)}{a_{yx}(t)}$$

Dynamical state mobility

- Onsager kinetic coefficients at the transition level:

$$m_{xy}(t) \coloneqq \frac{a_{xy}(t) - a_{yx}(t)}{\ln a_{xy}(t) - \ln a_{yx}(t)} = \frac{j_{xy}(t)}{f_{xy}(t)}$$

Dynamical state mobility

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Linear response regime	Nonlinear regime
$J_x = \sum_y \mu_{xy} F_y$	$j_{xy}(t) = m_{xy}(t) f_{xy}(t)$
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  - Kinetic cost $\mathcal{M}_\tau \coloneqq \int_0^\tau m_t \, dt = \tau \langle m \rangle_\tau$

Dynamical state mobility

- Analogy between the dynamical state mobility and macroscopic mobility

Macroscopic level	Microscopic level
$J = \mu F$	$j_{xy} = m_{xy} f_{xy}$
Einstein relation $\lvert F \rvert \ll 1$ $\mu = \beta D$	Einstein-like relation $\lvert f_{xy} \rvert \ll 1$ $m_{xy} = (a_{xy} + a_{yx})/2$

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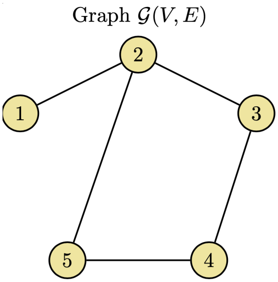
- Improved thermodynamic uncertainty relation [Gingrich et al., PRL (2016)]

$$\frac{\langle J \rangle^2}{\mathrm{Var}[J]} \leq \eta \frac{\Sigma_\tau}{2} \leq \frac{\Sigma_\tau}{2}$$

$\eta \coloneqq 2\mathcal{M}_\tau/\mathcal{A}_\tau \leq 1$

Wasserstein distance based on connectivity of Markov jump processes

- $\mathcal{G}(V, E)$: graph characterizing topology of Markov jump process
  - V: set of states
  - $(x, y) \in E$ if jump between x and y is allowed



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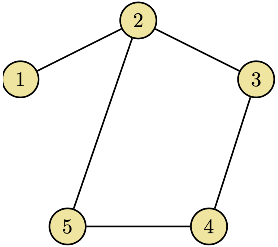
Graph $\mathcal{G}(V, E)$



Shortest-path distances $\{d_{xy}\}$

0	1	2	3	2
1	0	1	2	1
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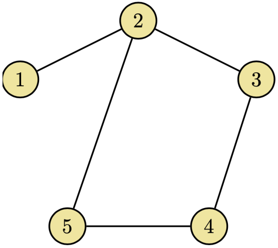

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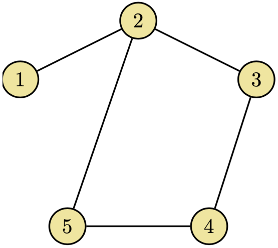

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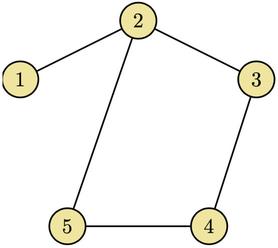

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Thermodynamic interpretation of discrete Wasserstein distances

Theorem 1

The Wasserstein distance based on a topology $\mathcal{G}(V,E)$ can be written in variational forms as

$$\mathcal{W}_1(p^A, p^B) = \min_{\mathsf{W}_t} \int_0^\tau \sqrt{\sigma_t m_t}\, dt$$
$$= \min_{\mathsf{W}_t} \sqrt{\Sigma_\tau \mathcal{M}_\tau}$$

the minimum is taken over all transition rate matrices $\{\mathsf{W}_t\}_{0\le t\le\tau}$ which satisfy the master equation with the boundary conditions $p_0 = p^A$ and $p_\tau = p^B$ and induce subgraphs of $\mathcal{G}(V,E)$ for all times

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Proof:

[step 1] Prove that $\mathcal{W}_1(p^A,p^B) \le \int_0^\tau \sqrt{\sigma_t m_t}\, dt \le \sqrt{\Sigma_\tau \mathcal{M}_\tau}$ holds for all admissible Markovian dynamics that transform p^A into p^B

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[step 2] Construct a specific process that attains the equality

Remarks of Theorem 1

- Analogous thermodynamic properties with the continuous L^2-Wasserstein distance

$$\mathcal{W}_1(p^A, p^B) = \min_{\mathsf{W}_t} \sqrt{\bar{D}\tau\Sigma_\tau} \quad \left[\leftrightarrow W_2(p^A, p^B) = \min_{F_t} \sqrt{D\tau\Sigma_\tau}\right]$$

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- Tradeoff between irreversibility and state mobility: $\Sigma_\tau \mathcal{M}_\tau \geq \mathcal{W}_1(p_0, p_\tau)^2$
  - Either the thermodynamic or kinetic cost must be sacrificed to achieve a feasible state transformation

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  - For one-dimensional nearest-neighbor topology (i.e., jump between x and y is admitted if and only if $|x - y| = 1$), $d_{xy} = |x - y|$ and

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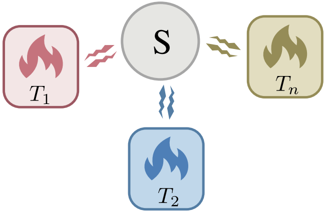
  - Taking the continuum limit yields

$$W_1(p^A, p^B) = \min_{j_t} \int_0^\tau \int_{\mathbb{R}} |j_t(x)| \, dx \, dt$$

  Providing a unified generalization of the Benamou–Brenier formula for the L^1-Wasserstein distance

Markovian open quantum dynamics

- Discrete-state dynamics obeying GKSL master equation
  $\dot{\varrho}_t = \mathcal{L}_t(\varrho_t) \coloneqq -i[H_t, \varrho_t] + \sum_k \mathcal{D}[L_k(t)]\varrho_t$



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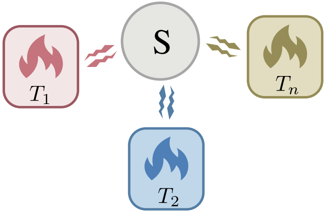
- Local detailed balance $L_k(t) = e^{s_k(t)/2} L_{k'}(t)^\dagger$
  $s_k(t) = -s_{k'}(t)$: entropy change in the environment



Entropy production, dynamical activity, and dynamical state mobility

- Irreversible entropy production

$$\Sigma_\tau \coloneqq \Delta S_{\rm sys} + \Delta S_{\rm env} \geq 0$$

$\Delta S_{\rm sys} \coloneqq S(\varrho_\tau) - S(\varrho_0)$: change in the von Neumann entropy
$\Delta S_{\rm env} \coloneqq \int_0^\tau \sum_k \operatorname{tr}\left\{L_k(t)\varrho_t L_k^\dagger(t)\right\} s_k(t)\, dt$: environmental entropy production

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- Dynamical state mobility

$$m_t \coloneqq \frac{1}{2}\sum_k e^{-s_k(t)/2}\left\langle L_k(t)^\dagger, [\![\varrho_t]\!]_{s_k(t)}(\mathcal{P}_t[L_k(t)^\dagger])\right\rangle$$

$\langle X, Y\rangle \coloneqq \mathrm{tr}\left\{X^\dagger Y\right\}$
$\mathcal{P}_t[X] \coloneqq X - \sum_x \langle x_t|X|x_t\rangle\, |x_t\rangle\!\langle x_t|$
$[\![\phi]\!]_\theta(X) \coloneqq e^{-\theta/2}\int_0^1 e^{\theta u}\phi^u X \phi^{1-u}\, du$
$\varrho_t = \sum_x p_x(t)\, |x_t\rangle\!\langle x_t|$: spectral decomposition of the density matrix ϱ_t

Quantum Wasserstein distance

- Naive quantum extension

$$W_q(\varrho^A, \varrho^B) \coloneqq \min_{\varrho^{AB} \in \Pi(\varrho^A, \varrho^B)} \mathrm{tr}\{C \varrho^{AB}\}$$

$\Pi(\varrho^A, \varrho^B)$: set of density matrices ϱ^{AB} satisfying $\mathrm{tr}_B\, \varrho^{AB} = \varrho^A$ and $\mathrm{tr}_A\, \varrho^{AB} = \varrho^B$
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- $W_q(\varrho_0, \varrho_\tau) > 0$ even for unitary dynamics $\varrho_\tau = U \varrho_0 U^\dagger$ with zero entropy production
  - Relating dissipation to the optimal transport distances defined in the naive form is impossible

Quantum Wasserstein distance

- Considering dissipative structure of Lindblad dynamics, we define
$$\mathcal{W}_q(\varrho^A, \varrho^B) \coloneqq \frac{1}{2} \min_{V^\dagger V = \mathbb{1}} \| V \varrho^A V^\dagger - \varrho^B \|_1$$
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  - Analytical expression

$$\mathcal{W}_q(\varrho^A, \varrho^B) = \frac{1}{2} \sum_x |p_x^A - p_x^B| = \mathcal{T}(p^A, p^B)$$

    $\{p_x^A\}$ and $\{p_x^B\}$: increasing eigenvalues of ϱ^A and ϱ^B, respectively

Thermodynamic interpretation of quantum Wasserstein distance

Theorem 2

The quantum Wasserstein distance can be written in the following variational form:

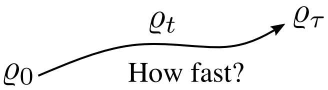
$$\mathcal{W}_q(\varrho^A, \varrho^B) = \min_{\mathcal{L}_t} \int_0^\tau \sqrt{\sigma_t m_t}\, dt$$
$$= \min_{\mathcal{L}_t} \sqrt{\Sigma_\tau \mathcal{M}_\tau}$$

the minimum is taken over all super-operators $\{\mathcal{L}_t\}_{0\leq t\leq\tau}$ that satisfy the Lindblad master equation with boundary conditions $\varrho_0 = \varrho^A$ and $\varrho_\tau = \varrho^B$

Applications

- **Thermodynamic speed limit**: lower bound on the operational time required for state transformations

$$\tau \geq \frac{\mathcal{W}_1(p_0, p_\tau)}{\left\langle\sqrt{\sigma m}\right\rangle_\tau} \geq \frac{\mathcal{W}_1(p_0, p_\tau)}{\sqrt{\langle\sigma\rangle_\tau \langle m\rangle_\tau}}$$



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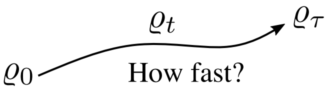
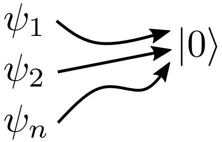


- **Finite-time Landauer principle**: lower bound on heat dissipation required for erasing information

$$Q \geq -T\Delta S_{\mathrm{sys}} + \frac{\mathcal{W}_1(p_0, p_\tau)^2}{\tau \beta \langle m \rangle_\tau}$$



Numerical demonstration

Pareto-optimal protocol in information erasure of qubit

$$\mathcal{F}_q[\{\varepsilon_t, \theta_t\}] \coloneqq \lambda Q - (1-\lambda) F(\varrho_\tau, \varrho_*)$$

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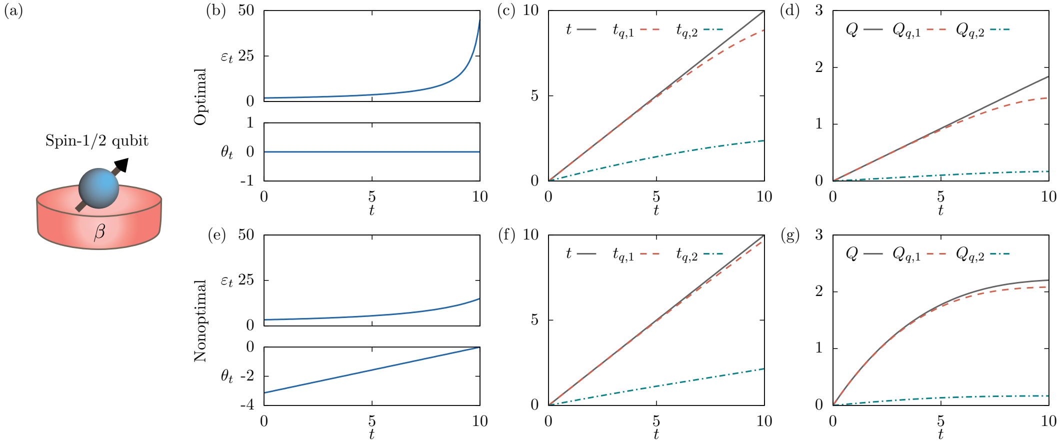


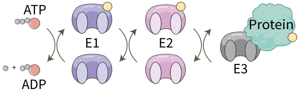
Take-home message

	Continuous	**Classical discrete**	**Quantum**
Wasserstein distance	$W_\alpha \ (\alpha \geq 1)$	$\mathcal{W}_1$	$\mathcal{W}_q$
Thermodynamic interpretation of optimal transport	Benamou–Brenier formula $W_2(p^A, p^B) = \min \sqrt{\tau D \Sigma_\tau}$ $W_1(p^A, p^B) \leq \min \sqrt{\tau D \Sigma_\tau}$	Theorem 1 $\mathcal{W}_1(p^A, p^B) = \min \sqrt{\tau \langle m \rangle_\tau \Sigma_\tau}$	Theorem 2 $\mathcal{W}_q(\varrho^A, \varrho^B) = \min \sqrt{\tau \langle m \rangle_\tau \Sigma_\tau}$
Minimum dissipation	$\min \Sigma_\tau = \dfrac{\mathcal{W}_2(p^A, p^B)^2}{\tau D}$	$\min_{\langle m \rangle_\tau = D} \Sigma_\tau = \dfrac{\mathcal{W}_1(p^A, p^B)^2}{\tau D}$	$\min_{\langle m \rangle_\tau = D} \Sigma_\tau = \dfrac{\mathcal{W}_q(\varrho^A, \varrho^B)^2}{\tau D}$
Thermodynamic speed limit	$\tau \geq \dfrac{W_{2(1)}(p^A, p^B)}{\sqrt{D \langle \sigma \rangle_\tau}}$	$\tau \geq \dfrac{\mathcal{W}_1(p^A, p^B)}{\sqrt{\langle m \rangle_\tau \langle \sigma \rangle_\tau}}$	$\tau \geq \dfrac{\mathcal{W}_q(\varrho^A, \varrho^B)}{\sqrt{\langle m \rangle_\tau \langle \sigma \rangle_\tau}}$

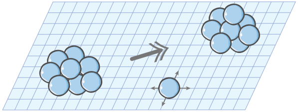
Optimal transport and speed limits

Motivation

- Interacting systems generally form spatial structures in their dynamics



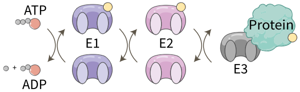
chemical reactions



bosonic transport

Motivation

- Interacting systems generally form spatial structures in their dynamics
  - Jump processs with dense connectivity may relax faster



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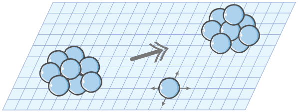

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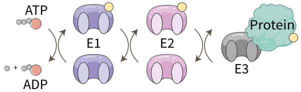

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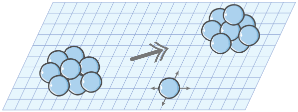

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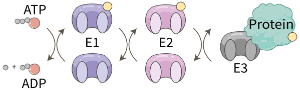
$$\tau \geq \frac{\mathcal{L}(\boldsymbol{x}_0, \boldsymbol{x}_\tau)}{\overline{v}}$$

$\mathcal{L}(\boldsymbol{x}_0, \boldsymbol{x}_\tau) \leq C$ (irrelevant to system size)

$\overline{v}$: velocity generally being order of system size



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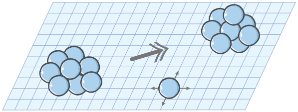
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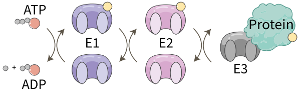

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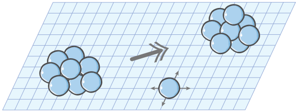

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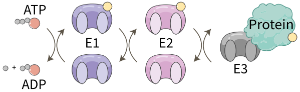
- Metrics that are scalable to system size should be considered



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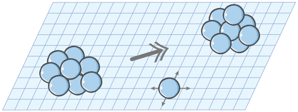

bosonic transport

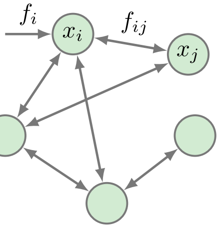
General dynamics

- A physical state $\boldsymbol{x}_t = [x_1(t), \ldots, x_N(t)]$ described by

$$\dot{x}_i(t) = f_i(t) + \sum_{j \in \mathcal{B}_i} f_{ij}(t)$$

$f_{ij}(t) = -f_{ji}(t)$: flow exchange between i and j

$f_i(t)$: arbitrary external flow



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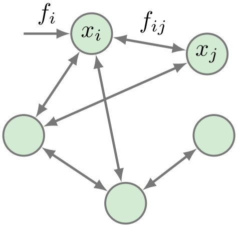
- Examples include probability distributions of discrete systems, reactant concentrations of chemical reaction networks, or physical observables in quantum systems



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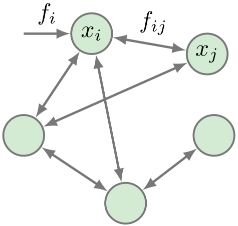


$$\dot{x}_i(t) = f_i(t) + \sum_{j \in \mathcal{B}_i} f_{ij}(t)$$

- Time-dependent velocity

$$v_{t,\lambda} \coloneqq \lambda \sum_i |f_i(t)| + \sum_{(i,j) \in \mathcal{E}} |f_{ij}(t)|$$

  $\lambda \geq 0$: weighting factor

General result

Speed limit using generalized Wasserstein distance

The operational time required for transform $\boldsymbol{x}_0$ into $\boldsymbol{x}_\tau$ is lower bounded by the Wasserstein distance divided by the average velocity:

$$\tau \geq \frac{\mathcal{W}_{1,\lambda}(\boldsymbol{x}_0, \boldsymbol{x}_\tau)}{\langle v_{t,\lambda} \rangle_\tau} \quad \forall \lambda \geq 0$$

In the case that the external flows are absent [i.e., $f_i(t) = 0$]

$$\tau \geq \frac{\mathcal{W}_1(\boldsymbol{x}_0, \boldsymbol{x}_\tau)}{\langle v_t \rangle_\tau}$$

Quantitative

Reaction channels

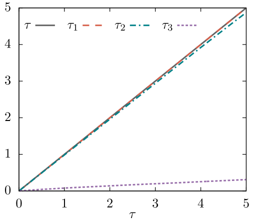
$$2X_1 \underset{k_b}{\overset{k_f}{\rightleftharpoons}} X_1 + X_2$$

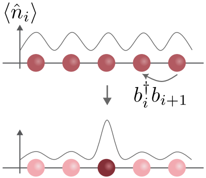
$$2X_2 \underset{k_b}{\overset{k_f}{\rightleftharpoons}} X_2 + X_3$$

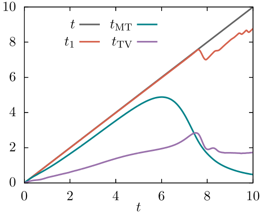
$$\vdots$$

$$2X_9 \underset{k_b}{\overset{k_f}{\rightleftharpoons}} X_9 + X_{10}$$

Graph $G(\mathcal{V}, \mathcal{E})$







$$H_t = -\gamma \sum_{i=1}^{N-1} (b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) + \sum_{i=1}^{N} U_i(t) \hat{n}_i (\hat{n}_i - 1)/2$$

Applications

Quantitative

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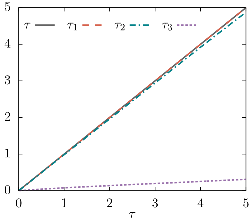

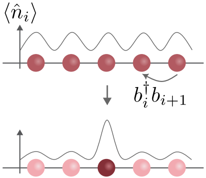

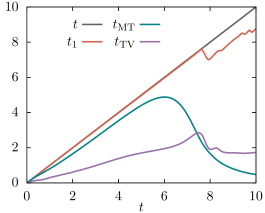

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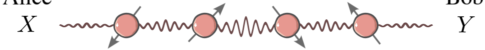
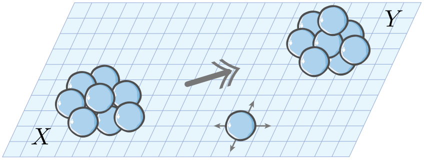
Qualitative

Quantum state transfer



Bosonic transport



time $\propto$ dist(X, Y)

Applications – Bosonic transport

- Model of bosons that hop on an arbitrary finite-dimensional lattice and interact with each other

$$H \coloneqq -\gamma \sum_{(i,j)} (b_i^\dagger b_j + b_j^\dagger b_i) + \sum_{Z \subseteq \Lambda} h_Z$$

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- Weakly coupled to a Markovian thermal reservoir and can exchange particles with the reservoir

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$L_{i,+} = \sqrt{\gamma_{i,+}} b_i^\dagger$ and $L_{i,-} = \sqrt{\gamma_{i,-}} b_i$: jump operators that characterize the absorption and emission of bosons at site i

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- Time evolution of $x_i(t)$ can be expressed using $f_i(t) = \mathrm{tr}\left\{L_{i,+} \varrho_t L_{i,+}^\dagger\right\} - \mathrm{tr}\left\{L_{i,-} \varrho_t L_{i,-}^\dagger\right\}$ and $f_{ij}(t) = 2\gamma \Im\left[\mathrm{tr}\left\{b_j^\dagger b_i \varrho_t\right\}\right]$

Applications – Bosonic transport

- Upper bound of velocity

$$v_{t,\lambda} \leq \gamma d_G \mathcal{N}_t + \lambda \frac{\sigma_t}{2} \Phi\left(\frac{\sigma_t}{2a_t}\right)^{-1}$$

d_G: maximal vertex degree

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  - This statement holds for *arbitrary* initial states, including the pure states considered in [Faupin et al., PRL (2022)]

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- Applicable to a wide range of dynamics, from classical to quantum, from continuous time to discrete time

Thermodynamic interpretation of discrete Wasserstein distances

Corollary 1

The discrete Wasserstein distance can be expressed in terms of irreversible entropy production and dynamical activity as

$$\mathcal{W}_1(p^A, p^B) = \min_{\mathsf{W}_t} \int_0^\tau \frac{\sigma_t}{2} \Phi\left(\frac{\sigma_t}{2a_t}\right)^{-1} dt$$
$$= \min_{\mathsf{W}_t} \frac{\Sigma_\tau}{2} \Phi\left(\frac{\Sigma_\tau}{2\mathcal{A}_\tau}\right)^{-1}$$

$\Phi(x)$: inverse function of $x \tanh(x)$

Thermodynamic interpretation of discrete Wasserstein distances

Corollary 2

The discrete Wasserstein distance can be expressed in terms of pseudo entropy production and dynamical activity as

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$$= \min_{\mathsf{W}_t} \sqrt{\Sigma_\tau^{\mathrm{ps}} \mathcal{A}_t}$$

$\sigma_t^{\mathrm{ps}} \coloneqq \dot{\Sigma}_t^{\mathrm{ps}}$: the pseudo entropy production rate

Thermodynamic interpretation of discrete Wasserstein distances

Corollary 2

The discrete Wasserstein distance can be expressed in terms of pseudo entropy production and dynamical activity as

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- Pseudo entropy production rate

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- $(\Sigma_\tau, \mathcal{M}_\tau)$ and $(\Sigma_\tau^{\mathrm{ps}}, \mathcal{A}_\tau)$ are two thermodynamic-kinetic conjugate pairs in the context of optimal transport

Thermodynamic interpretation of quantum Wasserstein distance

Corollary 3

The quantum Wasserstein distance can be expressed in terms of irreversible entropy production and dynamical activity as

$$\mathcal{W}_q(\varrho^A, \varrho^B) = \min_{\mathcal{L}_t} \int_0^\tau \frac{\sigma_t}{2} \Phi\left(\frac{\sigma_t}{2a_t}\right)^{-1} dt$$
$$= \min_{\mathcal{L}_t} \frac{\Sigma_\tau}{2} \Phi\left(\frac{\Sigma_\tau}{2\mathcal{A}_\tau}\right)^{-1}$$