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量子測定下の非ユニタリーダイナミクスの対称性とトポロジー

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arXiv: 2408.16974 (to appear in PRL)

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Outline

1. Introduction & Motivation

2. Topology of Monitored Quantum Dynamics

3. Universal Stochastic Equations of Monitored Quantum Dynamics

Monitored quantum dynamics

Unitary quantum dynamics

propagation of quantum correlations and entanglement

scrambling and/or thermalization

Quantum measurements

nonunitarity freezes quantum dynamics (i.e., quantum Zeno effect)

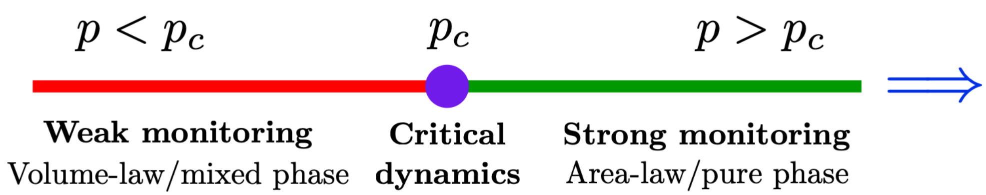
nonequilibrium steady states

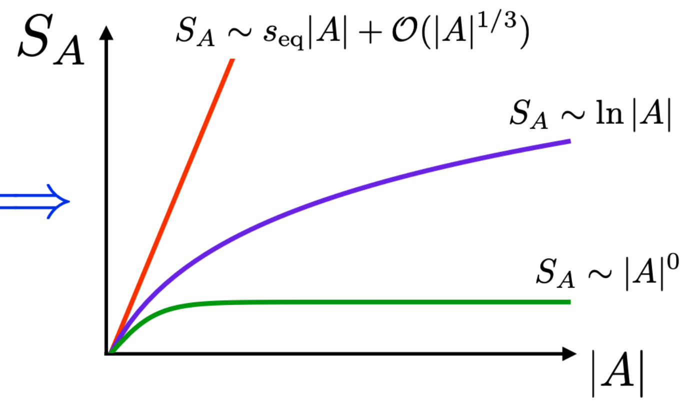
☆ Competition between unitary dynamics and measurements?

⟶ dynamical phase transitions unique to open quantum systems

Measurement-induced phase transitions

Skinner *et al.*, PRX **9**, 031009 (2019)
Li *et al.*, PRB **98**, 205136 (2018)





Fisher *et al.*, Annu. Rev. Condens. Matter Phys. **14**, 335 (2023)

Purification transitions: mixed vs pure phases

Gullans & Huse, PRX **10**, 041020 (2020)

☆ **New problems in statistical physics, quantum information science, etc.**

(connection with quantum error correction)

Anderson localization

Periodic crystals

electrons are delocalized through crystals (i.e., Bloch theorem)

ballistic/diffusive transport phenomena (i.e., metals)

Spatial disorder

Anderson, PR **109**, 1492 (1958)

localizes (electronic) waves (i.e., Anderson localization)

prevents thermalization and diffusion (i.e., insulators)

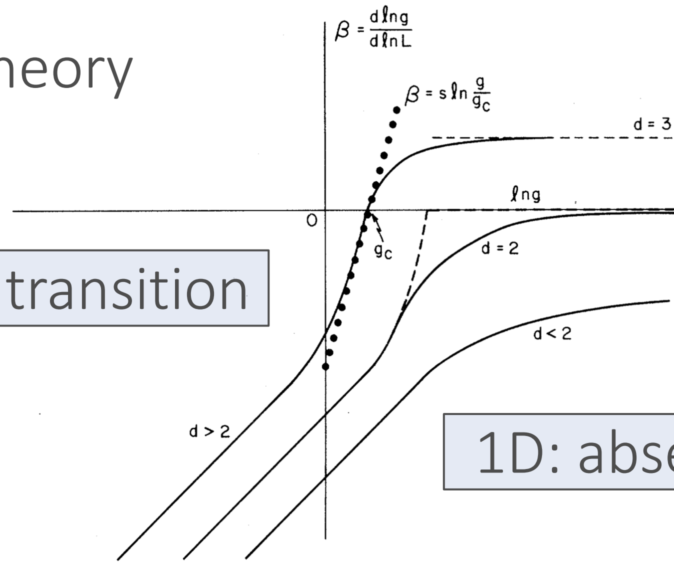
☆ **Competition between coherent dynamics and disorder?**

→ **phase transitions unique to disordered systems (i.e., Anderson transitions)**

Anderson transitions

Anderson transitions: localization transitions induced by disorder

scaling theory



Anderson, PR **109**, 1492 (1958)

Abrahams *et al.*, PRL **42**, 673 (1979)

3D: localization transition

1D: absence of delocalization

Universality classes of Anderson transitions are determined by

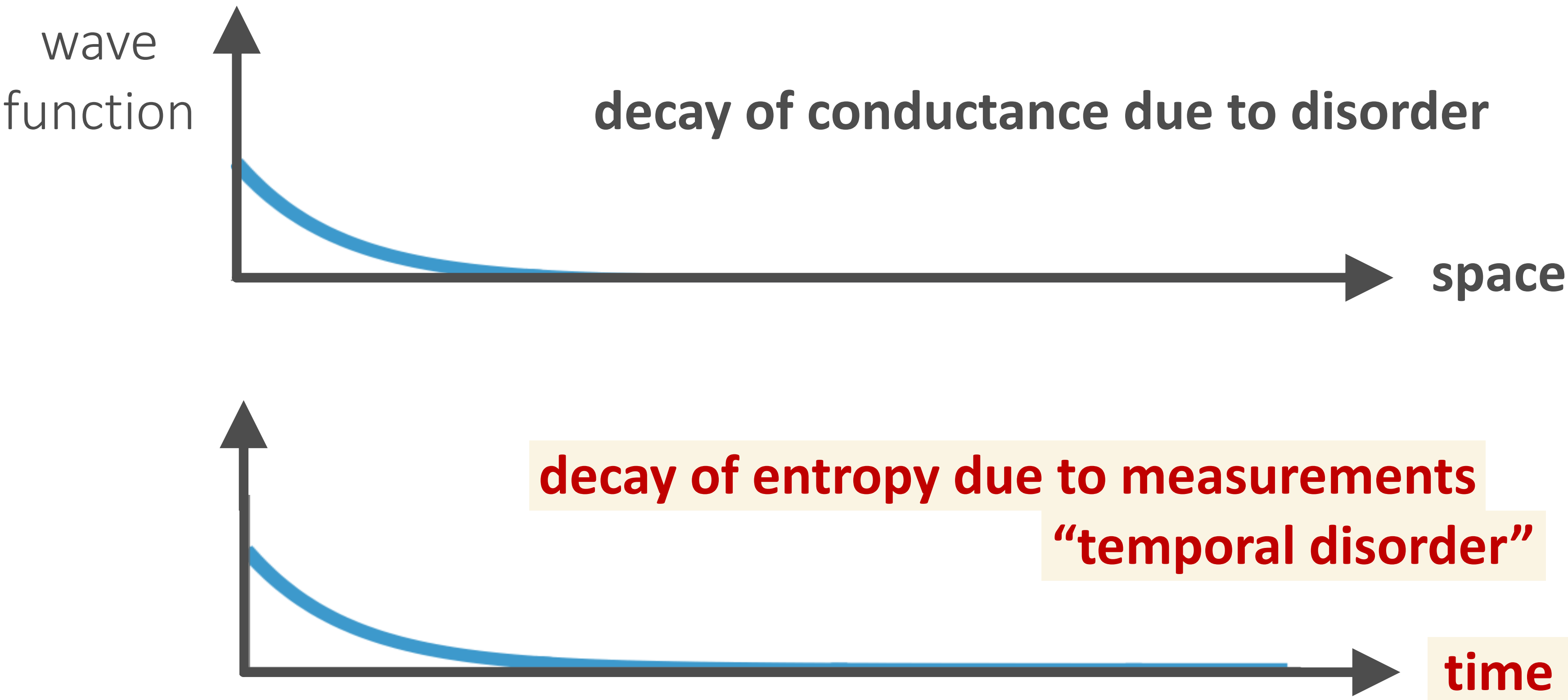
(1) Symmetry (especially, discrete internal symmetry)

(2) Spatial dimensions

(3) Topology (e.g., quantum Hall transitions)

Jian *et al.*, PRB **106**, 134206 (2022)

☆ Measurement-induced phase transitions for free fermions can be considered as Anderson transitions in spacetime!



Can we justify this connection and clarify differences?

Nonlinear sigma models

☆ **Both MIPT and AT are described by the same effective field theory.**
(for free fermions) **(nonlinear sigma model)**

$$S_n[Q] = \frac{1}{t} \sum_{\mu=x,t} \int dx dt \ \mathrm{tr}\left[(\partial_\mu Q^\dagger)(\partial_\mu Q)\right]$$

Jian *et al.*, arXiv:2302.09094
Fava *et al.*, PRX **13**, 041045 (2023)
Poboiko *et al.*, PRX **13**, 041046 (2023)

$$\begin{cases} \text{complex fermions} : Q \in \mathrm{U}\,(R) \to \text{NO transitions in (1+1)-D} \\ \text{Majorana fermions} : Q \in \mathrm{O}\,(R) \to \text{transitions in (1+1)-D} \end{cases}$$

→ **unique scaling of steady-state entanglement entropy**

$$S_\alpha \sim \frac{1+\alpha}{96\alpha} (\log L)^2$$

☆ **Different replica indices**

$$\text{MIPT} : R \to 1, \quad \text{AT} : R \to 0$$

different critical phenomena

Motivation

How can we connect the field theory description to microscopic models of monitored quantum dynamics?

$$S_n[Q] = \frac{1}{t} \sum_{\mu} \int d^d\boldsymbol{x}dt \ \mathrm{tr}\left[(\partial_\mu Q^\dagger)(\partial_\mu Q)\right] + \underbrace{(\text{topological terms})}_{?}$$

What are the roles of symmetry and topology in monitored quantum dynamics?

How can we classify universality classes of measurement-induced phase transitions?

Results (1)

We develop the tenfold classification of symmetry and topology for monitored free fermions.

We establish the bulk-boundary correspondence:
spacetime topology leads to anomalous boundary states.

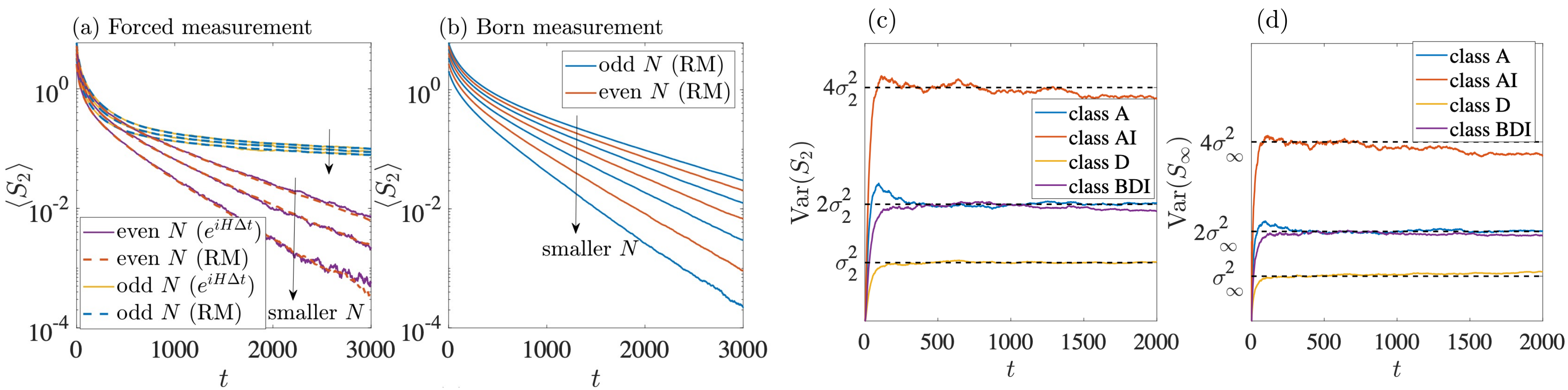
Class		$d+1=1$	$d+1=2$	$d+1=3$	$d+1=4$	$d+1=5$	$d+1=6$	$d+1=7$	$d+1=8$
A	$\mathcal{C}_1$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	$\mathcal{C}_0$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathcal{R}_1$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	$\mathcal{R}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	$\mathcal{R}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	$\mathcal{R}_5$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	$\mathcal{R}_6$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	$\mathcal{R}_7$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$\mathcal{R}_0$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Xiao and **Kawabata**, arXiv:2412.06133

Results (2)

We derive universal stochastic equations of monitored free fermions in 0+1 dimension.

We find universal purification dynamics and entropy fluctuations.



Xiao, Ohtsuki & **Kawabata**, arXiv:2408.16974
(to be published in PRL)

Topology of Monitored Quantum Dynamics

Xiao & **Kawabata**, arXiv:2412.06133

Altland-Zirnbauer symmetry

☆ **3-fold symmetry class by Wigner & Dyson**

Wigner (1959)
Dyson, J. Math. Phys. **3**, 1199 (1962)

time reversal $\underbrace{\mathcal{T}}_{\pm 1} H^* \mathcal{T}^{-1} = H$

anti-unitary
(with complex conjugation)

☆ **10-fold symmetry class by Altland & Zirnbauer**

Altland & Zirnbauer, PRB **55**, 1142 (1997)

particle hole $\mathcal{C} H^* \mathcal{C}^{-1} = -H$ anti-unitary

chiral (sublattice) $\Gamma H \Gamma^{-1} = -H$ unitary

- **Universality**

Random matrix theory, Anderson transitions, topological phases, ……

Topological insulators and superconductors

General and comprehensive theoretical framework of TIs and TSCs:

Periodic table based on spatial dimension and symmetry

AZ Symmetry				Dimension							
Class	TRS	PHS	CS	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$					
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$						
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$					
CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Quantum Hall insulator

Kitaev/Majorana chain

Quantum spin Hall insulator

Schnyder, Ryu, Furusaki & Ludwig, PRB **78**, 195125 (2008)

Kitaev, AIP Conf. Proc. **1134**, 22 (2009)

Kraus operators

Generic quantum operation (CPTP map)

$$\rho \longmapsto \rho' = \sum_i K_i \rho \underbrace{K_i^\dagger}_{\text{Kraus operator}}$$

Open quantum dynamics (Markovian)

$$\rho(t) = \sum_{i_1, i_2, \cdots, i_n} K_{i_n} \cdots K_{i_1} \rho_0 K_{i_1}^\dagger \cdots K_{i_n}^\dagger$$

This is the average of "measurement outcomes" $(i_1, \cdots, i_n)$

quantum trajectory $K_{i_n} \cdots K_{i_1} \rho K_{i_1}^\dagger \cdots K_{i_n}^\dagger$

Quantum trajectory

Quantum trajectory $|\psi_0\rangle \longmapsto |\psi_t\rangle \propto \underbrace{K_t K_{t-\Delta t} \cdots K_{\Delta t}}_{=:\, K_{[0,t]}} |\psi_0\rangle$

Kraus operators incorporate both random unitary evolution and stochastic nonunitary measurements.

– Born measurements $\left(\| K_{[0,t]} |\psi_0\rangle \|^2\right)$

dynamics depends on measurement probabilities at each time

– Forced measurements

dynamics evolves according to prior (postselected) probabilities

NOTE: Replica limit for nonlinear sigma models:

Born: $R \to 1$, Forced: $R \to 0$

Jian *et al.*, arXiv:2302.09094
Fava *et al.*, PRX **13**, 041045 (2023)
Poboiko *et al.*, PRX **13**, 041046 (2023)

Example

Purification dynamics of Gaussian mixed states of *N* complex fermions:

– Unitary dynamics $U_t \in \mathrm{U}\left(N\right)$

– Continuous measurement of the particle number n_i

$$M_t = \mathrm{diag}\left(e^{\epsilon_i}\right)$$

$$\epsilon_i = \begin{cases} \left(2\left\langle n_i \right\rangle_t - 1\right)\gamma dt + \sqrt{\gamma} dW_t^i & \text{(Born measurement)} \\ \underbrace{\sqrt{\gamma}}_{\text{measurement strength}} \underbrace{dW_t^i}_{\text{Wiener process}} & \text{(forced measurement)} \end{cases}$$

measurement strength Wiener process $\langle dW_t^i \rangle = 0, \langle dW_t^i dW_t^j \rangle = \delta_{ij} dt$

→ **cumulative Kraus operators (single-particle quantum trajectory)**

$$K_{[0,t]} = \left(M_t U_t\right) \cdots \left(M_{\Delta t} U_{\Delta t}\right)$$

Non-Hermitian dynamical generators

Over an infinitesimal time interval [t, t+Δt]

$$|\psi_{t+\Delta t}\rangle \propto K_t\,|\psi_t\rangle$$

→ **Stochastic Schrödinger equation**

$$L_t\,|\psi_t\rangle = 0, \quad L_t := \partial_t - H_t, \quad e^{H_t \Delta t} := K_t$$

effective non-Hermitian "Hamiltonian"

☆ **The open quantum dynamics is encoded in K_t or L_t**

Relationship with disordered electrons

Monitored dynamics	Kraus operators K	"Hamiltonian" L
Disordered electrons	transfer matrix T	Hamiltonian H

Single-particle Schrödinger equation of disordered electrons

$$\begin{pmatrix} \psi_{x+1} \\ \psi_x \end{pmatrix} = T_x \begin{pmatrix} \psi_x \\ \psi_{x-1} \end{pmatrix}$$

Kramer *et al.*, Int. J. Mod. Phys. B **24**, 1841 (2010)

Localization length is quantified by $\left\| \prod_{x=1}^{L} T_x \right\| \sim e^{-L/\xi}$

☆ **Kraus operators *K$_t$*: transfer matrices in the temporal direction**

$$|\psi_{t+\Delta t}\rangle \propto K_t |\psi_t\rangle$$

Purification timescale is quantified by $\left\| \prod_{t}^{T} K_t \right\| \sim e^{-T/\tau}$

***L$_t$* serves as an effective non-Hermitian Hamiltonian**

Symmetry of Kraus operators (1)

Kraus operators inherently incorporate spacetime randomness.

spatial disorder & temporal noise (intrinsic to quantum measurements)

→ **Symmetry preserved by the product of Kraus operators is only relevant to the monitored quantum dynamics.**

$$\left(K_{[0,t]} := K_t K_{t-\Delta t} \cdots K_{\Delta t} \right)$$

$$\mathcal{T} K_t^* \mathcal{T}^{-1} = K_t \quad \left(\mathcal{T}\mathcal{T}^* = \pm 1 \right)$$

$$\mathcal{C} \left(K_t^T \right)^{-1} \mathcal{C}^{-1} = K_t \quad \left(\mathcal{C}\mathcal{C}^* = \pm 1 \right)$$

$$\Gamma \left(K_t^\dagger \right)^{-1} \Gamma^{-1} = K_t \quad \left(\Gamma^2 = 1 \right)$$

Symmetry of Kraus operators (2)

- **Complex conjugation is preserved for the product of *K_t***

$$\mathcal{T} K_t^* \mathcal{T}^{-1} = K_t \longrightarrow \mathcal{T} K_{[0,t]}^* \mathcal{T}^{-1} = K_{[0,t]}$$

$$\left(K_{[0,t]} := K_t K_{t-\Delta t} \cdots K_{\Delta t}\right)$$

- **Transposition/inversion is NOT preserved for the product of *K_t***

$$K_t^{T/-1} = K_t \longrightarrow K_{[0,t]}^{T/-1} = \underline{K_{\Delta t} \cdots K_{t-\Delta t} K_t} \neq K_{[0,t]}$$

Temporal direction is reversed

- **Combination of transposition and inversion is preserved**

$$\mathcal{C}\,(K_t^T)^{-1}\mathcal{C}^{-1} = K_t$$

$$\Gamma\,(K_t^\dagger)^{-1}\Gamma^{-1} = K_t$$

Symmetry of dynamical generators

Symmetry of Kraus operators:

$$\mathcal{T} K_t^* \mathcal{T}^{-1} = K_t \quad (\mathcal{T}\mathcal{T}^* = \pm 1)$$
$$\mathcal{C} \left(K_t^T \right)^{-1} \mathcal{C}^{-1} = K_t \quad (\mathcal{C}\mathcal{C}^* = \pm 1)$$
$$\Gamma \left(K_t^\dagger \right)^{-1} \Gamma^{-1} = K_t \quad (\Gamma^2 = 1)$$

→ **Symmetry of effective dynamical generators:**

cf. **Kawabata** *et al.*, PRX **9**, 041015 (2019)

$$\mathcal{T} L_t^* \mathcal{T}^{-1} = L_t \quad (\mathcal{T}\mathcal{T}^* = \pm 1)$$ "time-reversal symmetry"

$$\mathcal{C} L_t^T \mathcal{C}^{-1} = -L_t \quad (\mathcal{C}\mathcal{C}^* = \pm 1)$$ "particle-hole symmetry"

$$\Gamma L_t^\dagger \Gamma^{-1} = -L_t \quad (\Gamma^2 = 1)$$ "chiral symmetry"

☆ **Tenfold internal symmetry classes of monitored dynamics**

☆ Tenfold symmetry classification of *K* and *L*

(single-particle Kraus operators & associated dynamical generators)

(noncompact type)

Class	TRS $\mathcal{T}$	PHS $\mathcal{C}$	CS Γ	Classifying space (K)	Classifying space (L)
A	0	0	0	$\mathrm{GL}(N,\mathbb{C})/\mathrm{U}(N) \cong \mathcal{C}_1$	$\mathcal{C}_1$ (AIII)
AIII	0	0	1	$\mathrm{U}(N,N)/\mathrm{U}(N)\times\mathrm{U}(N) \cong \mathcal{C}_0$	$\mathcal{C}_0$ (A)
AI	$+1$	0	0	$\mathrm{GL}(N,\mathbb{R})/\mathrm{O}(N) \cong \mathcal{R}_7$	$\mathcal{R}_1$ (BDI)
BDI	$+1$	$+1$	1	$\mathrm{O}(N,N)/\mathrm{O}(N)\times\mathrm{O}(N) \cong \mathcal{R}_0$	$\mathcal{R}_2$ (D)
D	0	$+1$	0	$\mathrm{O}(N,\mathbb{C})/\mathrm{O}(N) \cong \mathcal{R}_1$	$\mathcal{R}_3$ (DIII)
DIII	-1	$+1$	1	$\mathrm{O}^*(2N)/\mathrm{U}(N) \cong \mathcal{R}_2$	$\mathcal{R}_4$ (AII)
AII	-1	0	0	$\mathrm{U}^*(2N)/\mathrm{Sp}(N) \cong \mathcal{R}_3$	$\mathcal{R}_5$ (CII)
CII	-1	-1	1	$\mathrm{Sp}(N,N)/\mathrm{Sp}(N)\times\mathrm{Sp}(N) \cong \mathcal{R}_4$	$\mathcal{R}_6$ (C)
C	0	-1	0	$\mathrm{Sp}(N,\mathbb{C})/\mathrm{Sp}(N) \cong \mathcal{R}_5$	$\mathcal{R}_7$ (CI)
CI	$+1$	-1	1	$\mathrm{Sp}(N,\mathbb{R})/\mathrm{U}(N) \cong \mathcal{R}_6$	$\mathcal{R}_0$ (AI)

Xiao and **Kawabata**, arXiv:2412.06133

cf. tensor-network formulation: Jian, Bauer, Keselman & Ludwig, PRB **106**, 134206 (2022)

Time-reversal symmetry

Physical time-reversal symmetry: $\mathcal{T} K_t^* \mathcal{T}^{-1} = K_{\underline{-t}}$

time reversal

<u>NOT</u> exactly respected due to temporal noise

(may be respected on average, though)

“time-reversal symmetry” more relevant to monitored dynamics:

$$\mathcal{T} K_t^* \mathcal{T}^{-1} = K_{\underline{t}}$$

Behaves as “internal symmetry” in spacetime

Topology

Topology is captured by homotopy groups of classifying spaces

$$\pi_0\left(\underline{\mathcal{C}_{s-(d+1)}}\right), \pi_0\left(\underline{\mathcal{R}_{s-(d+1)}}\right)$$

Classifying spaces (determined solely by symmetry)

Connection with point-gap topology

Gong *et al.*, PRX **8**, 031079 (2018)

Kawabata *et al.*, PRX **9**, 041015 (2019)

$$\tilde{L}_t := \begin{pmatrix} 0 & L_t \\ L_t^\dagger & 0 \end{pmatrix}$$

Non-Hermitian topology of L_t = Hermitian topology of $\tilde{L}_t$

☆ Topological classification of non-Hermitian dynamical generators *L*

(spacetime dimensions)

Class		$d+1=1$	$d+1=2$	$d+1=3$	$d+1=4$	$d+1=5$	$d+1=6$	$d+1=7$	$d+1=8$
A	$\mathcal{C}_1$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	$\mathcal{C}_0$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathcal{R}_1$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	$\mathcal{R}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	$\mathcal{R}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	$\mathcal{R}_5$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	$\mathcal{R}_6$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	$\mathcal{R}_7$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$\mathcal{R}_0$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Xiao and **Kawabata**, arXiv:2412.06133

- **Bott periodicity in *K*-theory**

  2: complex classes (A, AIII)

  8: real classes (AI, BDI, D, DIII, AII, CII, C, CI)

Steady-state topology

Topological classification of L_t in d+1 dimensions

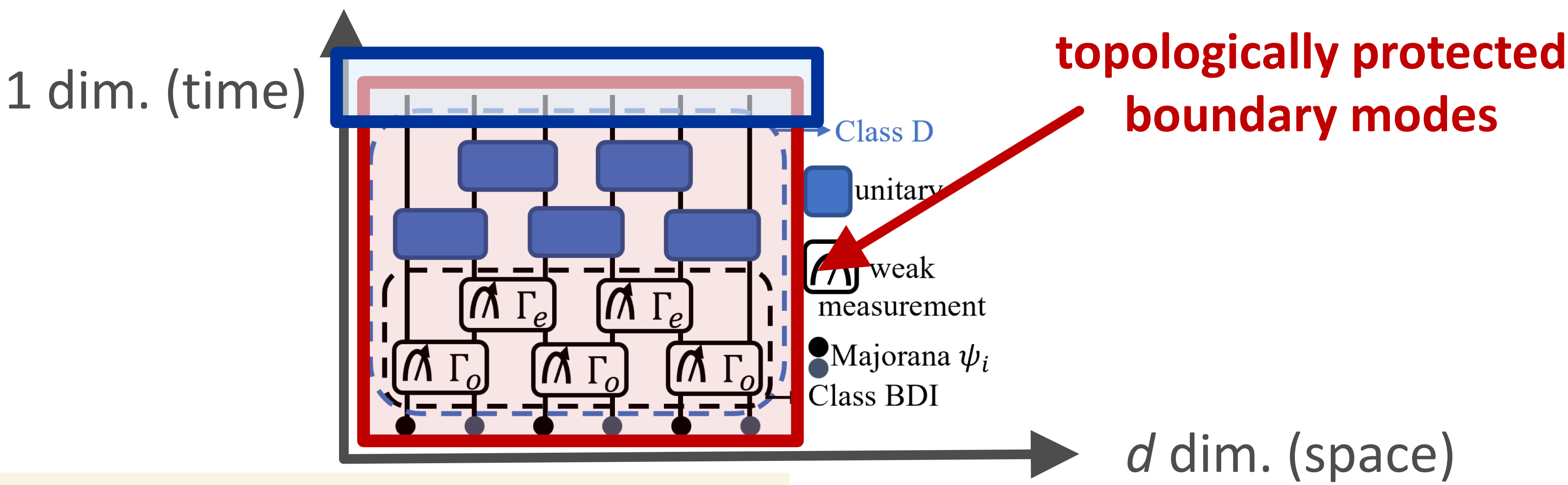
= Topological classification of Hermitian systems in d dimensions

(ensured by dimensional reduction in K-theory)

Appendix E in **Kawabata** *et al.*, PRX **9**, 041015 (2019)

☆ Correspondence of spacetime topology and steady-state topology

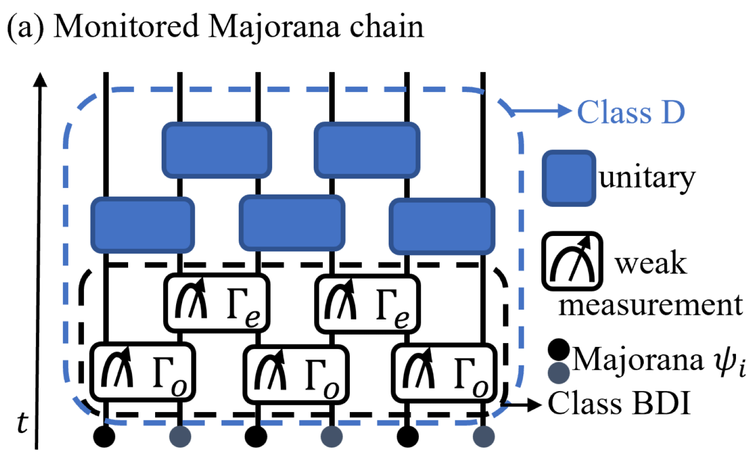
***d*-dim. space topology of steady state**



(d+1)-dim. spacetime topology of L_t

(1+1)-D classes D & BDI

Monitored Majorana fermions in one spatial dimension



Nahum *et al.*, PRR **2**, 023288 (2020)
Merritt *et al.*, PRB **107**, 064303 (2023)
Fava *et al.*, PRX **13**, 041045 (2023)

– measurements of fermion parity

$$\hat{K}_{2i-1,\pm} \propto e^{\pm i\Gamma\,(1+\Delta)\,\hat{\psi}_{2i-1}\hat{\psi}_{2i}/2}$$

$$\hat{K}_{2i,\pm} \propto e^{\pm i\Gamma\,(1-\Delta)\,\hat{\psi}_{2i}\hat{\psi}_{2i+1}/2}$$

– random unitary dynamics

$$\hat{U}_{2i-1} = e^{\theta_{2i-1}\hat{\psi}_{2i-1}\hat{\psi}_{2i}/2}$$

$$\hat{U}_{2i} = e^{\theta_{2i}\hat{\psi}_{2i}\hat{\psi}_{2i+1}/2}$$

Without unitary dynamics: class BDI (particle-hole & chiral)

With unitary dynamics: class D (particle-hole)

inherent in Majorana fermions

(Chern number of L_t = 1D winding number of H_t)

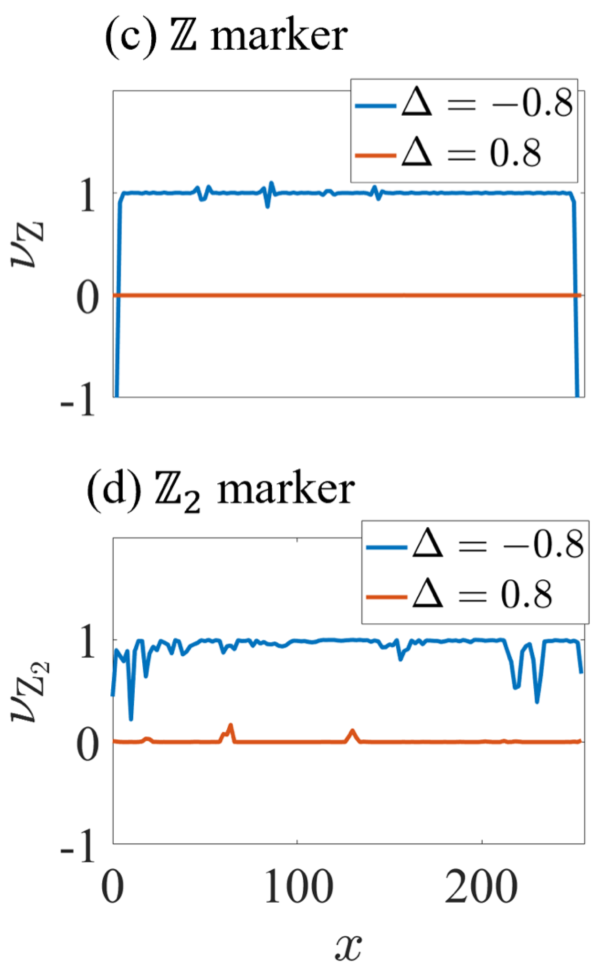
Measurement-only dynamics

Class		$d+1=1$	$d+1=2$	$d+1=3$	$d+1=4$	$d+1=5$	$d+1=6$	$d+1=7$	$d+1=8$
A	$\mathcal{C}_1$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	$\mathcal{C}_0$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathcal{R}_1$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	$\mathcal{R}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	$\mathcal{R}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	$\mathcal{R}_5$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	$\mathcal{R}_6$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	$\mathcal{R}_7$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$\mathcal{R}_0$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Class		$d+1=1$	$d+1=2$	$d+1=3$	$d+1=4$	$d+1=5$	$d+1=6$	$d+1=7$	$d+1=8$
A	$\mathcal{C}_1$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	$\mathcal{C}_0$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathcal{R}_1$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	$\mathcal{R}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	$\mathcal{R}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	$\mathcal{R}_5$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	$\mathcal{R}_6$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	$\mathcal{R}_7$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$\mathcal{R}_0$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Measurements with unitary dynamics

Topological invariants



$$\begin{cases} \mathbb{Z} \text{ topological invariant} & \text{(class BDI)} \\ \mathbb{Z}_2 \text{ topological invariant} & \text{(class D)} \end{cases}$$

Quantization of local topological marker

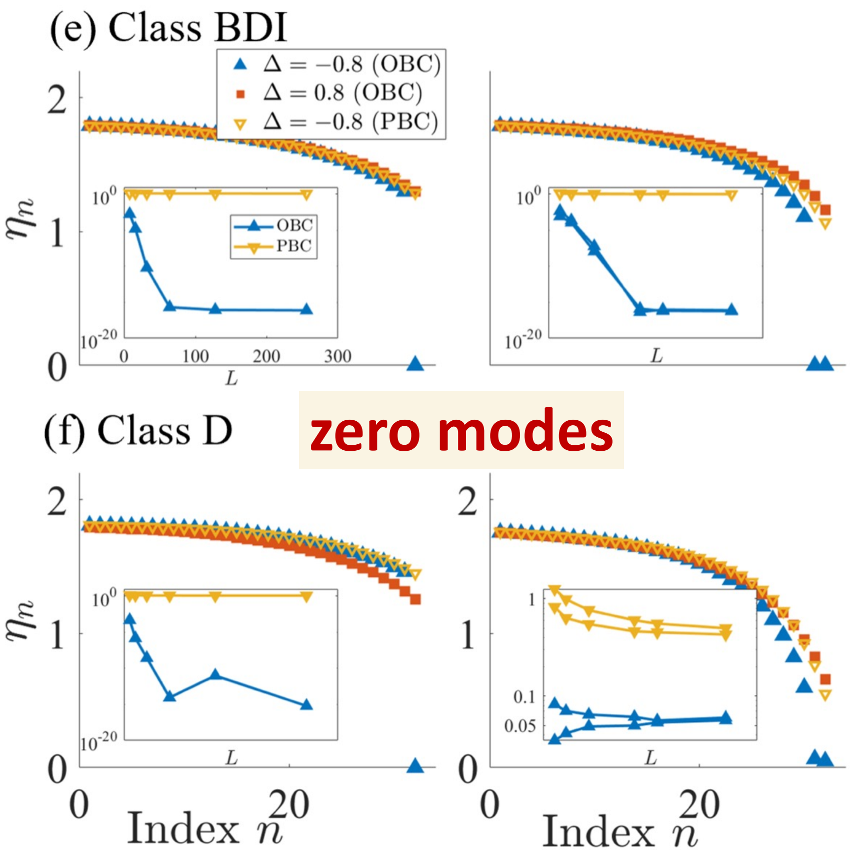
$$\nu = \begin{cases} 1 & (\Delta < 0) \\ 0 & (\Delta > 0) \end{cases}$$

Mondragon-Shem *et al.*, PRL **113**, 046802 (2014)
Hannukainen *et al.*, PRL **129**, 277601 (2022)

Zero modes

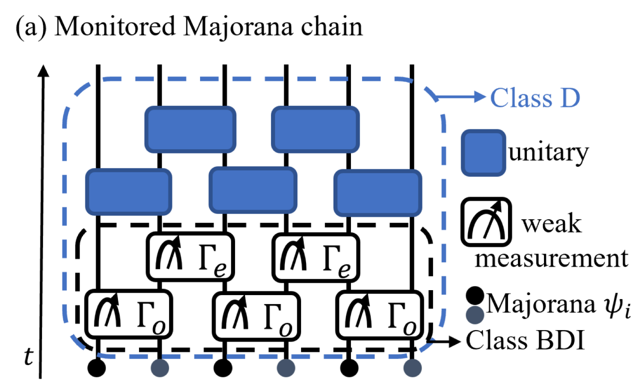
☆ **Topology leads to zero modes in the singular-value spectrum!**

ηt: logarithm of singular values of $K_{[0,\, t]}$



Majoranas at edges are isolated in the topological phase

Topologically protected slow purification (not exponential but algebraic)



☆ **Zero modes are ensured by spacetime topology of L_t**

Topological phase transitions

☆ **Topology is the origin of the measurement-induced phase transition**

Perturbative expansion of the nonlinear sigma model for class BDI

$$\beta\left(t\right) = d - 1 - 4t^{3} + \mathcal{O}\left(t^{4}\right) \qquad \left(t \geq 0\right)$$

Hikami, Phys. Lett. B **98**, 208 (1981)
Wegner, Nucl. Phys. B **316**, 663 (1989)

$$\longrightarrow \quad \beta < 0 \qquad \left(d \leq 1\right)$$

No phase transitions for 1+1 dimensions

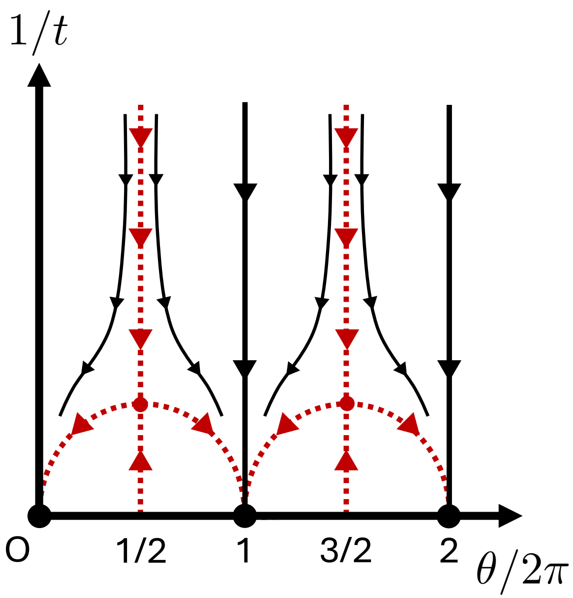
However, numerical simulations of lattice models demonstrate the measurement-induced phase transition.

→ **It requires a topological term!**

Nahum *et al.*, PRR **2**, 023288 (2020)
Merritt *et al.*, PRB **107**, 064303 (2023)
Fava *et al.*, PRX **13**, 041045 (2023)

Topological θ term

☆ Z-classified topology: θ term in the nonlinear sigma model



cf. Chen, **Kawabata**, Kulkarni & Ryu, PRB **111**, 054203 (2025)

$$S_n[Q] = \frac{1}{t} \sum_{\mu=x,t} \int dx dt \ \mathrm{tr}\left[(\partial_\mu Q^\dagger)(\partial_\mu Q)\right] + \underbrace{\theta N[Q]}_{\text{topological } \theta \text{ term}}$$

$$N[Q] := \sum_{\mu,\nu=x,t} \varepsilon^{\mu\nu} \int \frac{dx dt}{16\pi} \mathrm{tr}\left[Q(\partial_\mu Q)(\partial_\nu Q)\right]$$

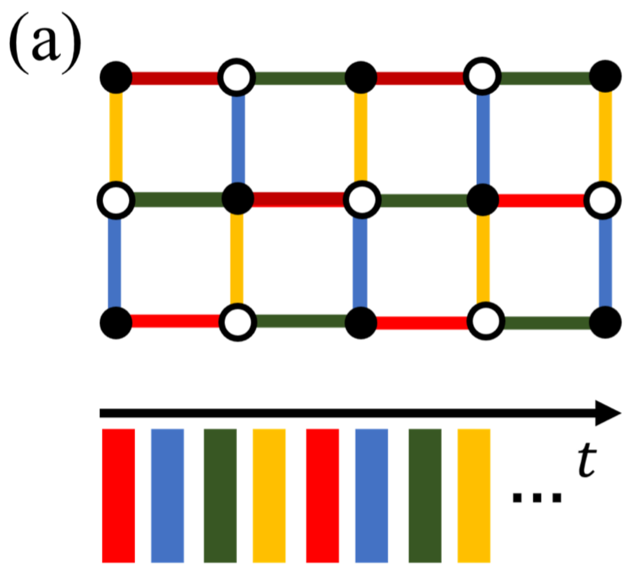
$$Q \in \underbrace{\mathrm{O}(2)/\mathrm{U}(1)}_{\text{specified solely by symmetry (class BDI)}}$$

☆ Analog of quantum Hall transitions in monitored dynamics!

Khmel'nitskii, JETP Lett. **38**, 552 (1983); Pruisken, PRL **61**, 1297 (1988)

(2+1)-D class A

Monitored complex fermions in two spatial dimensions



– unitary dynamics

$$\hat{U} = e^{\mathrm{i}\theta t_{\boldsymbol{r}\boldsymbol{r}'}\,(\hat{c}^\dagger_{\boldsymbol{r}}\hat{c}_{\boldsymbol{r}'}+\hat{c}^\dagger_{\boldsymbol{r}'}\hat{c}_{\boldsymbol{r}})}$$

$$t_{\boldsymbol{r}+\boldsymbol{e}_x} = t, \quad t_{\boldsymbol{r}+\boldsymbol{e}_y} = t\,(-1)^x$$

(Harper-Hofstadter Hamiltonian)

Proc. Phys. Soc. A **68**, 874 (1955)

– measurements

$$\hat{K}_{d\pm} \propto e^{\pm\Gamma\,(\hat{d}^\dagger\hat{d}-1/2)}, \quad \hat{K}_{f\pm} \propto e^{\pm\Gamma\,(\hat{f}^\dagger\hat{f}-1/2)}$$

$$\hat{d} = (\hat{c}_{\boldsymbol{r}} + \hat{c}_{\boldsymbol{r}'})/\sqrt{2}, \quad \hat{f} = (\hat{c}_{\boldsymbol{r}} - \hat{c}_{\boldsymbol{r}'})/\sqrt{2}$$

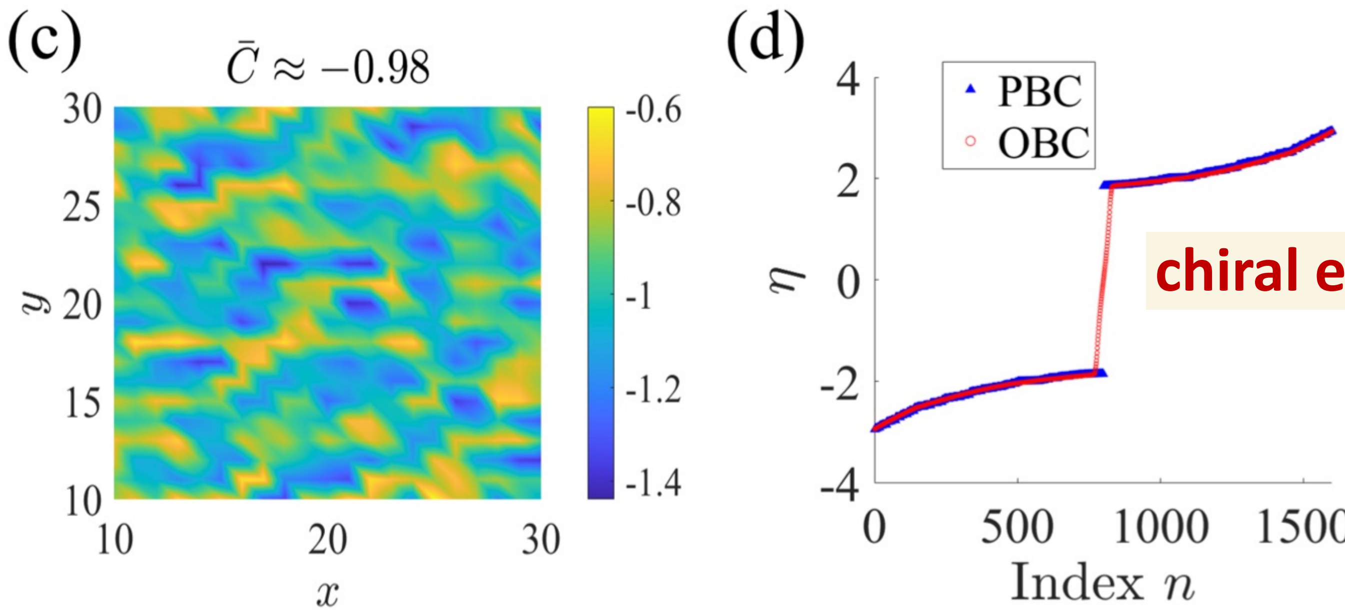
(3D winding number of L_t = 2D Chern number of H_t)

Class		$d+1=1$	$d+1=2$	$d+1=3$	$d+1=4$	$d+1=5$	$d+1=6$	$d+1=7$	$d+1=8$
A	$\mathcal{C}_1$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	$\mathcal{C}_0$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathcal{R}_1$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	$\mathcal{R}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	$\mathcal{R}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	$\mathcal{R}_5$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	$\mathcal{R}_6$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	$\mathcal{R}_7$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$\mathcal{R}_0$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Chiral edge modes

☆ **Spacetime topology leads to chiral edge modes!**

quantized local Chern marker



chiral edge modes

topologically protected slow purification

(forced measurements)

Can we analytically study universality classes of monitored dynamics?

It seems difficult in higher dimensions ......

→ **We derive the universal Fokker-Planck equations for monitored quantum dynamics in 0+1 dimension.**

(1) Universal purification dynamics (long time)

(2) Universal entropy fluctuations (short time)

Xiao, Ohtsuki & **Kawabata**, arXiv:2408.16974
(to be published in PRL)

Universal Stochastic Equations of Monitored Quantum Dynamics

Xiao, Ohtsuki & **Kawabata**, arXiv:2408.16974

(to be published in Phys. Rev. Lett.)

Monitored quantum dynamics (1)

Purification dynamics of Gaussian mixed states of *N* complex fermions:

– Unitary dynamics $U_t \in \mathrm{U}\left(N\right)$

– Continuous measurement of the particle number n_i

$$M_t = \mathrm{diag}\left(e^{\epsilon_i}\right)$$

$$\epsilon_i = \begin{cases} \left(2\left\langle n_i \right\rangle_t - 1\right)\gamma dt + \sqrt{\gamma} dW_t^i & \text{(Born measurement)} \\ \underbrace{\sqrt{\gamma}}_{\text{measurement strength}} \underbrace{dW_t^i}_{\text{Wiener process}} & \text{(forced measurement)} \end{cases}$$

measurement strength Wiener process $\langle dW_t^i \rangle = 0, \langle dW_t^i dW_t^j \rangle = \delta_{ij} dt$

→ **cumulative Kraus operators (single-particle quantum trajectory)**

$$K_{[0,t]} = \left(M_t U_t\right) \cdots \left(M_{\Delta t} U_{\Delta t}\right)$$

Monitored quantum dynamics (2)

Let us prepare the initial state as the completely mixed state $\rho_0 \propto 1$ and consider the decay of entropy (i.e., purification).

The time-evolved mixed state is determined by the Kraus operator:

$$\hat{\rho}_t \propto e^{2\hat{c}^\dagger P \hat{c}}, \qquad e^{2P} := K_{[0,t]} K_{[0,t]}^\dagger$$

– Two-point correlation function: $\langle \hat{c}_i^\dagger \hat{c}_j \rangle_t = \frac{1}{2} \left(\tanh P^T + 1 \right)_{ij}$

– αth Rényi entropy: $S_\alpha := \frac{1}{1-\alpha} \log \mathrm{Tr} \left(\frac{\hat{\rho}_t}{\mathrm{Tr}\, \hat{\rho}_t} \right)^\alpha = \sum_{n=1}^{N} f_{s\alpha}(z_n)$

$$\left(f_{s\alpha}(z_n) := \frac{1}{1-\alpha} \log \left[\frac{1}{(1+e^{2z})^\alpha} + \frac{1}{(1+e^{-2z})^\alpha} \right] \right)$$

Statistical evolution of z_n (eigenvalues of P) are relevant!

Fokker-Planck equation (1)

☆ **We derive the Fokker-Planck equations for** $p\left(\{z_n\};t\right)$

(probability distribution function for z_n's)

We perturbatively evaluate an incremental change of $p\left(\{z_n\};t\right)$

in the infinitesimal interval $[t, t+\Delta t]$

(functional renormalization group)

We model the unitary dynamics as the Haar-random unitary.

– try to capture universal chaotic features

(to be numerically confirmed for local lattice models)

– correspond to nonlinear sigma models in 0+1 dimension

– analytical tractability (random matrix theory)

Fokker-Planck equation (2)

Fokker-Planck equation for density-matrix spectra

$$\underbrace{\frac{N+1}{\gamma}\frac{\partial p}{\partial t} = -\sum_{n=1}^{N}\frac{\partial\left[\left(\mu_n+\nu_n\right)p\right]}{\partial z_n}}_{\text{drift term}} + \frac{1}{2}\sum_{m,n=1}^{N}\frac{\partial^2\left[\left(1+\delta_{mn}\right)p\right]}{\partial z_n \partial z_m}$$

$$\mu_n = \sum_{m\neq n}\coth\left(z_n - z_m\right)$$ (generic for spectra of random operators)

$$\nu_n = \begin{cases} 0 & \text{(forced measurement)} \\ \underbrace{\sum_m \left(1+\delta_{mn}\right)\tanh z_m}_{\text{positive feedback effect of measurement}} & \text{(Born measurement)} \end{cases}$$

- Counterpart in disordered electrons: DMPK equation

Dorokhov, JETP Lett. **36**, 318 (1982); Mello, Pereyra & Kumar, Ann. Phys. **181**, 290 (1988)

Exact solutions

Initial condition: completely mixed state $\rho_0 \propto 1$

Exact solution for forced measurements:

$$p_F(\{z_n\};t) = \mathcal{N}(t)\left(\prod_{n<m}(z_n - z_m)\sinh(z_n - z_m)\right)$$
$$\times \exp\left(-\frac{N+1}{2\gamma t}\sum_{n,m} z_n\left(-\frac{1}{N+1} + \delta_{nm}\right) z_m\right)$$

Exact solution for Born measurements:

$$p_B(\{z_n\};t) = e^{-N\gamma t/2}\underbrace{\left(\prod_n \cosh z_n\right)}\, p_F(\{z_n\};t)$$

Born's rule $\propto \mathrm{Tr}\,\hat{\rho}_t$

Purification dynamics

Long-time dynamics: $S_\alpha \sim \frac{\alpha}{\alpha - 1} \sum_{i=1}^{N} e^{-2\,|z_i|} \propto e^{-2\,\min_i |z_i|}$

→ purification time $\frac{1}{\tau_P} = 2 \lim_{t\to\infty} \frac{\min_i |\langle z \rangle_i|}{t}$

- **Born measurement**

$$\langle z_n \rangle \simeq \frac{2\,(n-l) - 1 + \mathrm{sgn}\,(n - l - 0^+)}{N+1} \gamma t \qquad (l = 0, 1, \cdots, N)$$

→ $\tau_P = \frac{N+1}{4\gamma}$

exponential decay of entropy due to measurements

Even-odd effect

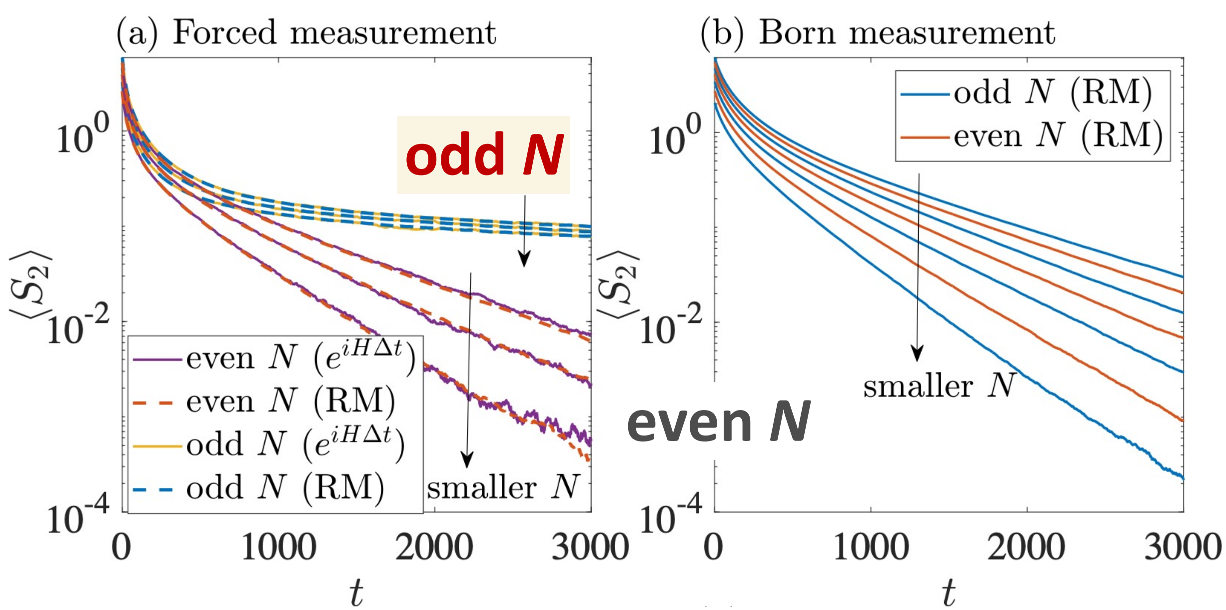
- **Forced measurement: exponential/algebraic purification**

$$\langle z_n \rangle = \frac{2n - N - 1}{N+1}\gamma t \qquad \min_n \frac{|\langle z_n \rangle|}{t} = \begin{cases} 0 & (\text{odd } N) \\ \gamma/(N+1) & (\text{even } N) \end{cases}$$

$$\longrightarrow \quad \tau_P = \begin{cases} \infty & (\text{odd } N) \\ 2(N+1)/\gamma & (\text{even } N) \end{cases}$$

algebraic decay

exponential decay



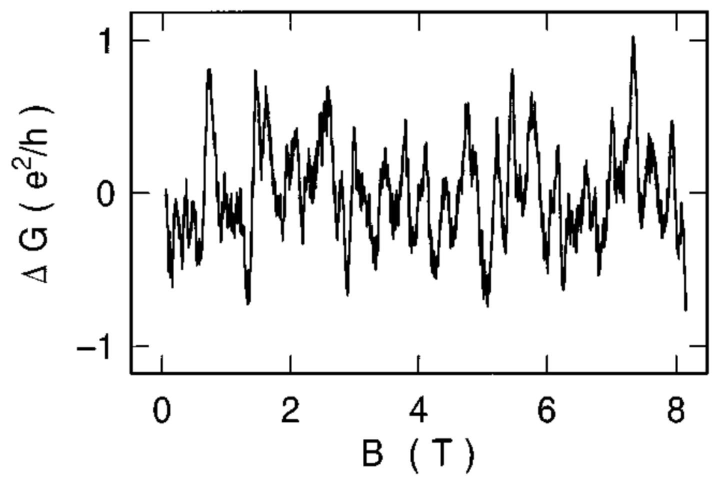
Universal conductance fluctuations

Universal conductance fluctuations in mesoscopic physics

$$\mathrm{Var}\left(\frac{G}{e^2/h}\right) = \frac{\mathcal{O}\,(1)\;\mathrm{const.}}{\beta}$$

$$\beta = 1, 2, 4\,(\text{time-reversal symmetry})$$

Unique quantum phenomenon in the diffusive regime



Washburn & Webb, Adv. Phys. **35**, 375 (1986)

☆ **Analog of UCF in monitored quantum dynamics?**

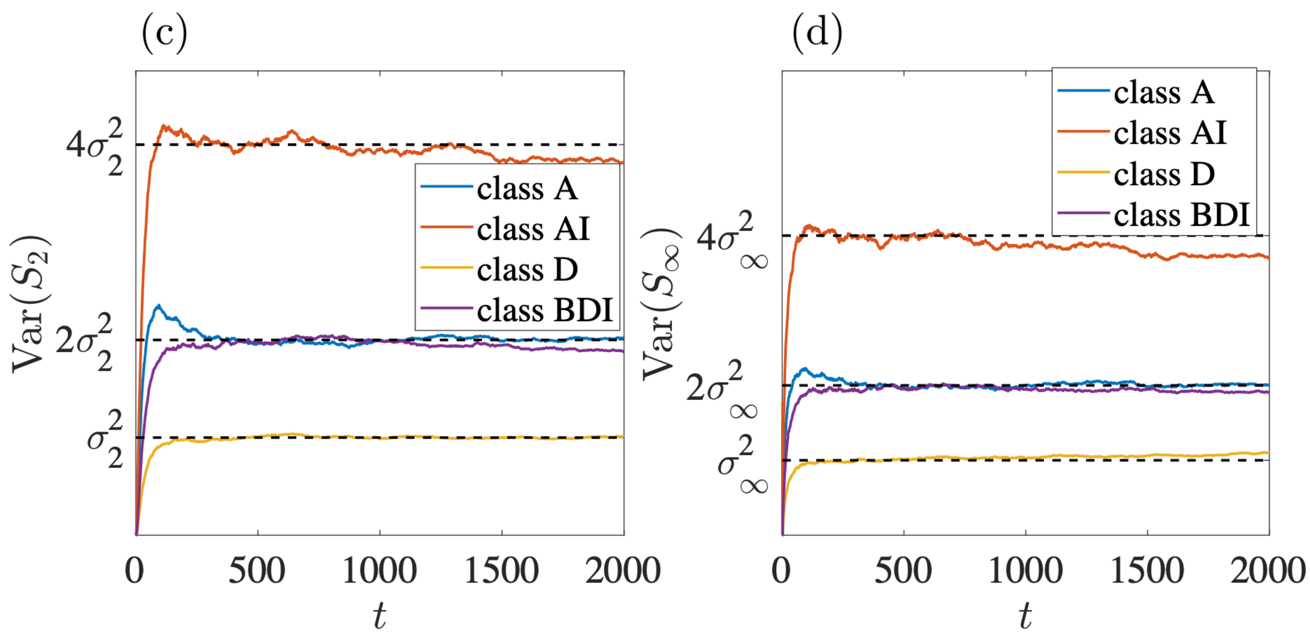
→ **We find universal entropy fluctuations!**

Universal entropy fluctuations

☆ **Universal entropy fluctuations in the large-*N* regime** $1 \ll \gamma t \ll N$
(short-time regime)

$$\mathrm{Var}\left(S_2\right) = 10\log 2 - 6\log\pi = 0.06309\cdots$$ (generalized to arbitrary α)

Applicable to both Born and forced measurements, even with locality



$$(\gamma = 0.04, N = 200)$$

Monitored Majorana fermions

☆ **Symmetry changes the universal Fokker-Planck equations**

- **Monitored Majorana fermions**

particle-hole symmetry: $(K_t^T)^{-1} = K_t, \quad L_t^T = -L_t$
(class D)

Majorana fermions (class D)

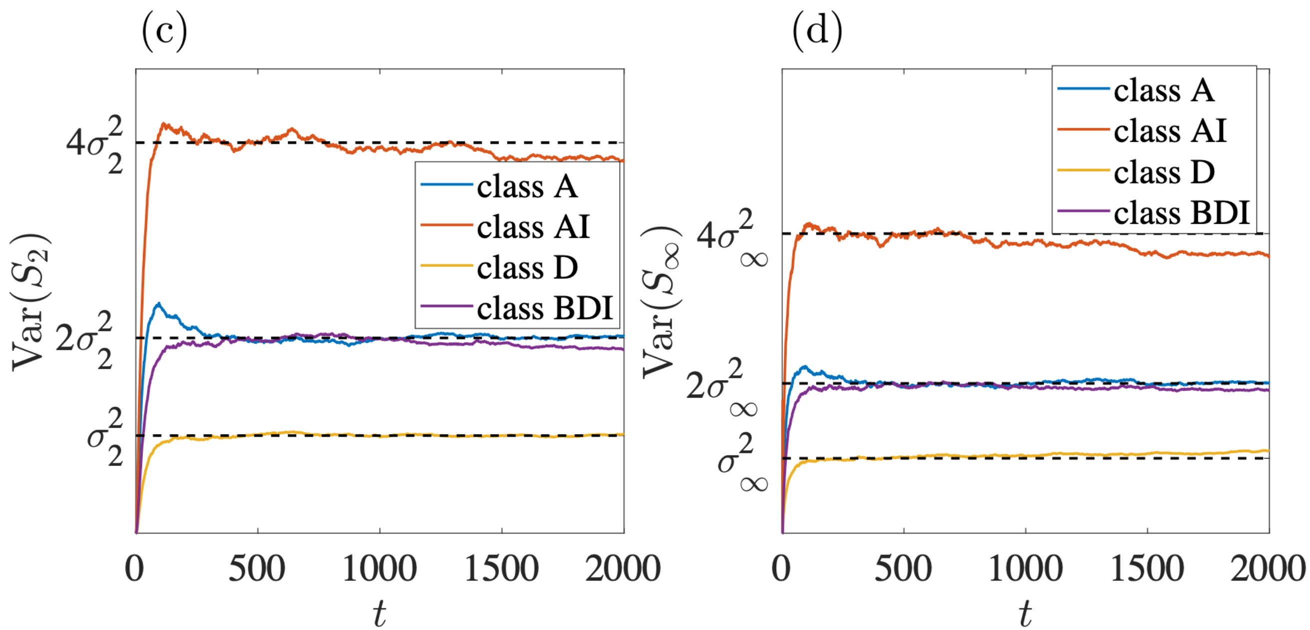
$$\begin{cases} \mu_n = \sum_{m\neq n} \left(\coth\left(z_n - z_m\right) + \coth\left(z_n + z_m\right)\right) \\ \nu_n = \tanh z_n \end{cases}$$

complex fermions (class A)

$$\begin{cases} \mu_n = \sum_{m\neq n} \coth\left(z_n - z_m\right) \\ \nu_n = \tanh z_n + \sum_m \tanh z_m \end{cases}$$

Symmetry classification

☆ **Universal entropy fluctuations provide a characteristic indicator of symmetry in the monitored dynamics!**



U(1)	H_t	$M_{0:t}$	L_{eff}	H_{dis}	$\mathrm{Var}(S_\alpha)$
$\surd$	A	$\mathrm{GL}(N,\mathbb{C})/\mathrm{U}(N)$	A	AIII	$2\sigma_\alpha^2$
$\surd$	D	$\mathrm{GL}(N,\mathbb{R})/\mathrm{O}(N)$	AI	BDI	$4\sigma_\alpha^2$
$\times$	D	$\mathrm{SO}(2N,\mathbb{C})/\mathrm{O}(2N)$	D	DIII	σ_α^2
$\times$	D$\oplus$D	$\mathrm{O}(N,N)/\mathrm{O}(N)\times\mathrm{O}(N)$	BDI	D	$2\sigma_\alpha^2$

Summary arXiv: 2408.16974 & 2412.06133

- We develop the tenfold classification of symmetry and topology for monitored free fermions and establish the bulk-boundary correspondence.

- We derive universal stochastic equations of monitored free fermions and find universal purification dynamics and entropy fluctuations.

Class		$d+1=1$	$d+1=2$	$d+1=3$	$d+1=4$	$d+1=5$	$d+1=6$	$d+1=7$	$d+1=8$
A	$\mathcal{C}_1$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	$\mathcal{C}_0$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathcal{R}_1$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	$\mathcal{R}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	$\mathcal{R}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	$\mathcal{R}_5$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	$\mathcal{R}_6$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	$\mathcal{R}_7$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$\mathcal{R}_0$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$