観測下の量子系における リャプノフスペクトル解析 と観測誘起相転移

Ken Mochizuki and Ryusuke Hamazaki,
Physical Review Letters **134**, 010410 (2025).

ERATO 沙川情報エネルギー変換プロジェクト

科研費 KAKENHI

Outline

- Introduction:
  - ▶ entanglement, spectrum, phase transitions
  - ▶ isolated systems **without randomness**
  - ▶ **random** dynamics in monitored systems
- Motivation and rough sketch of results
- Our results:
  - ▶ measurement-induced spectral transition
  - ▶ ground-state entanglement transition
- Foundation of the Lyapunov spectral analysis
- Summary

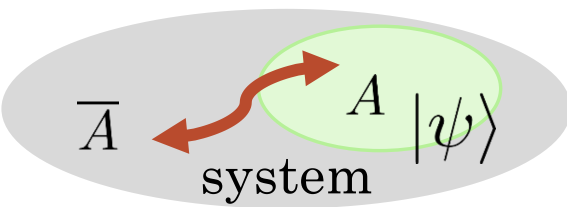
Entanglement of quantum states

$|\psi\rangle$: entangled? to what extent?

two spins: $|\psi\rangle = |\uparrow_1 \downarrow_2\rangle$: not entangled $\quad S_1(\psi) = 0$

$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1 \downarrow_2\rangle + |\downarrow_1 \uparrow_2\rangle)$: entangled $\quad S_1(\psi) = \log(2) \neq 0$

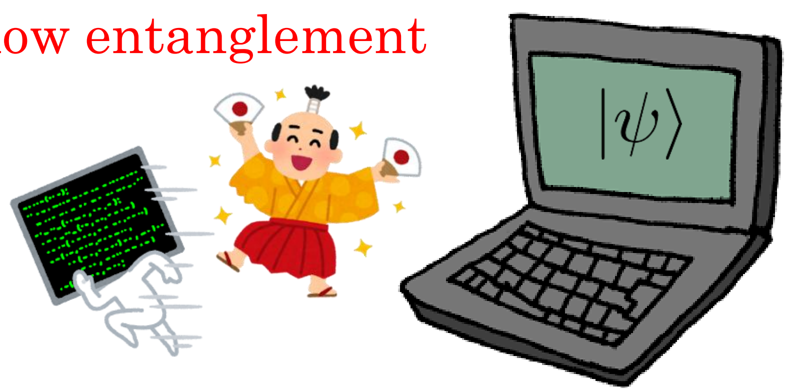
$\boxed{\text{entanglement entropy: } S_A(\psi) = -\mathrm{tr}_A\left(\rho_A(\psi)\log\left[\rho_A(\psi)\right]\right)}$



$\rho_A(\psi) = \mathrm{tr}_{\bar{A}}\left(|\psi\rangle\langle\psi|\right)$

large $S_A(\psi) \Rightarrow A$ and $\overline{A}$ are highly entangled

low entanglement

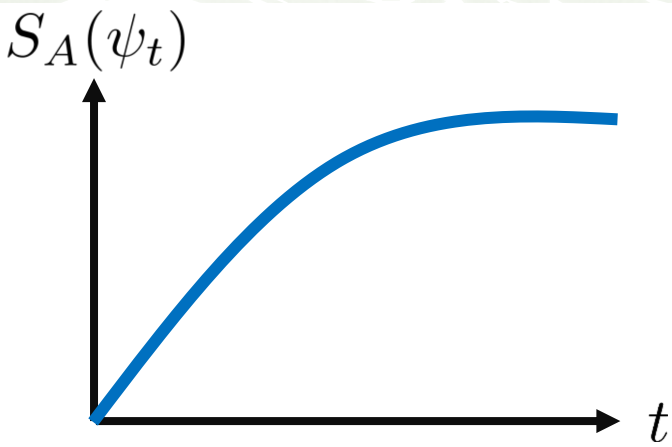


highly entangled

many spins on 1D chain:

$\cdots$

Dynamics in isolated quantum systems



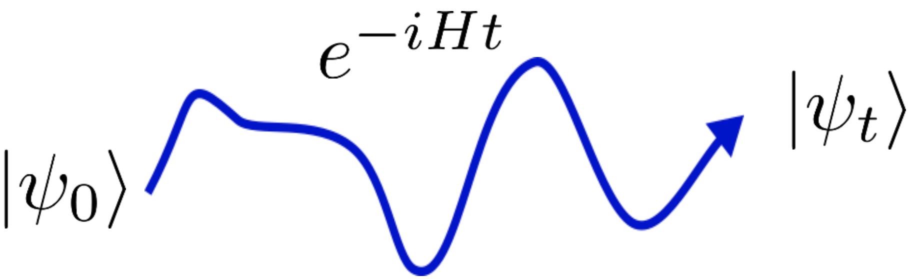
Hamiltonian dynamics

⬇

entanglement growth

$$|\psi_t\rangle = e^{-iHt}\,|\psi_0\rangle = \sum_i c_i e^{-i\varepsilon_i t}\,|\Psi_i\rangle$$

e.g. $H = \sum_x J\sigma_x^1\sigma_{x+1}^1 + B\sigma_x^3$

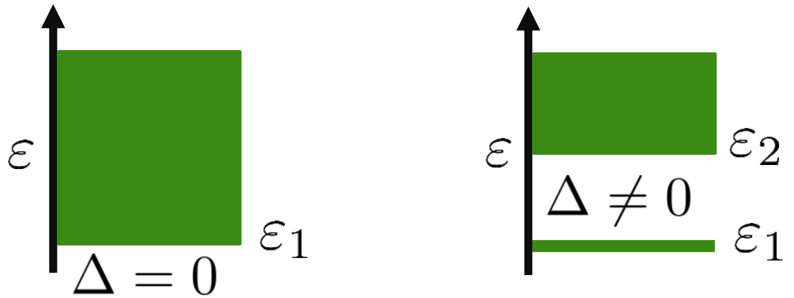


$$H\,|\Psi_i\rangle = \varepsilon_i\,|\Psi_i\rangle$$

spectral analysis
⇒ behavior of dynamics

Ground-state entanglement in isolated quantum systems

the spectral gap $\Delta = \varepsilon_2 - \varepsilon_1, \quad \varepsilon_i \leq \varepsilon_{i+1}$
transition of the gap $\leftrightarrow$ **ground-state phase transition**



e.g. XXZ model

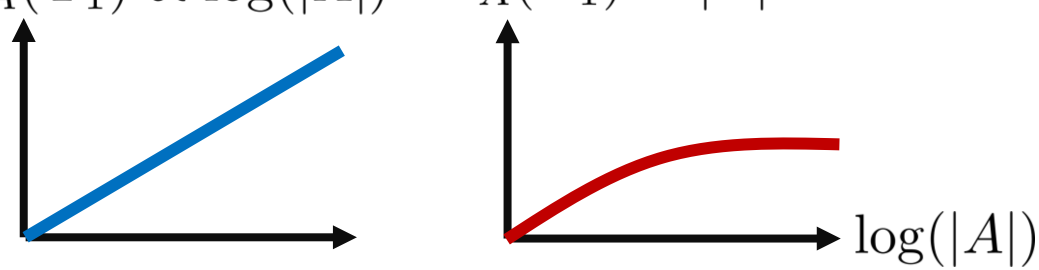
$$H = \sum_x \sigma_x^1 \sigma_{x+1}^1 + \sigma_x^2 \sigma_{x+1}^2 + J \sigma_x^3 \sigma_{x+1}^3$$

$$H \left|\Psi_i\right\rangle = \varepsilon_i \left|\Psi_i\right\rangle$$

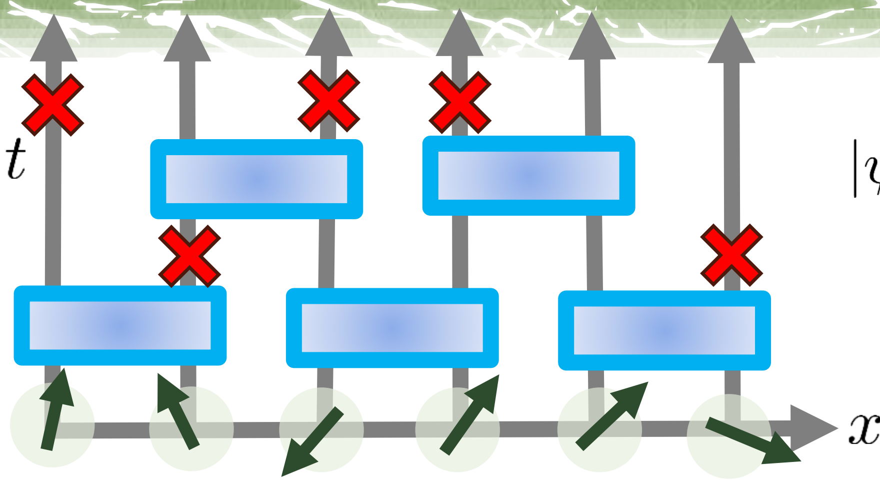
gapless phase **logarithmic-law**	**gapped phase** **area-law**

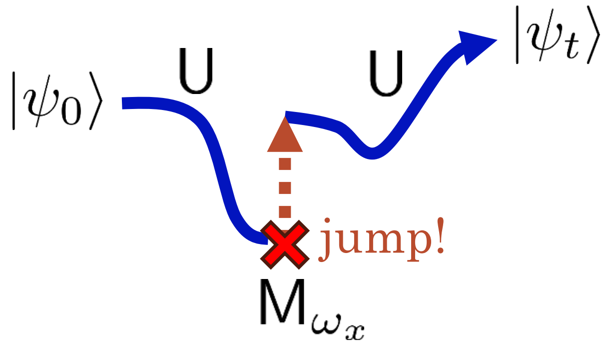
parameter J

$S_A(\Psi_1) \propto \log(|A|)$ $\qquad$ $S_A(\Psi_1) \propto |A|^0$

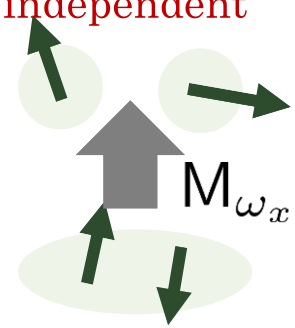


Monitored random dynamics





measurement at x with probability q $\quad |\psi_t\rangle \to \dfrac{\mathsf{M}_{\omega_x}|\psi_t\rangle}{\sqrt{P_{\psi_t}(\omega_x)}}$



$\omega_x = \uparrow\downarrow$: **random** measurement outcome

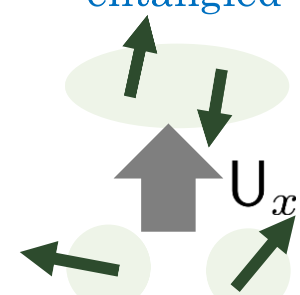
e.g. projective measurement $\mathsf{M}_{\omega_x} = |\omega_x\rangle\langle\omega_x|$ $\quad \mathsf{M}_{\uparrow_1}\dfrac{|\uparrow_1\downarrow_2\rangle + |\downarrow_1\uparrow_2\rangle}{\sqrt{2}} \propto |\uparrow_1\downarrow_2\rangle$

Born probability: $P_{\psi_t}(\omega_x) = \langle\psi_t|\mathsf{M}^\dagger_{\omega_x}\mathsf{M}_{\omega_x}|\psi_t\rangle$

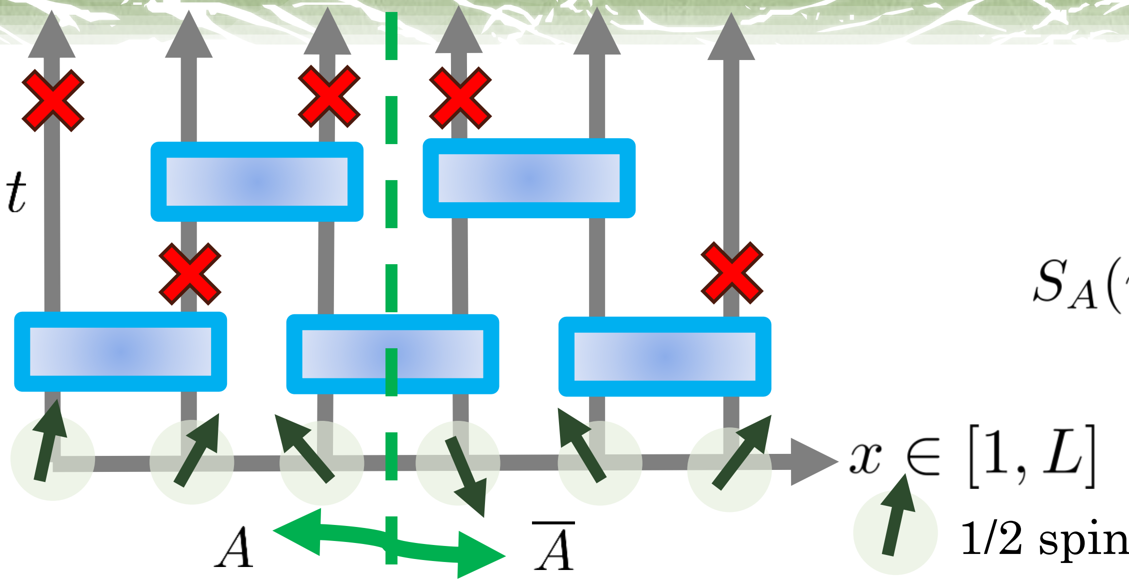

unitary gate $\quad |\psi_t\rangle \to \mathsf{U}_x|\psi_t\rangle$

c.f. Hamiltonian dynamics: $|\psi_t\rangle = e^{-iHt}|\psi_0\rangle$

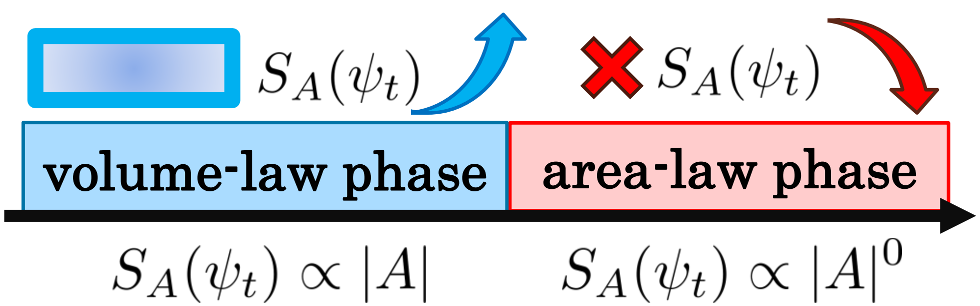
$$e^{-iH\delta t} \simeq e^{-iH_{\rm even}\delta t}e^{-iH_{\rm odd}\delta t}$$



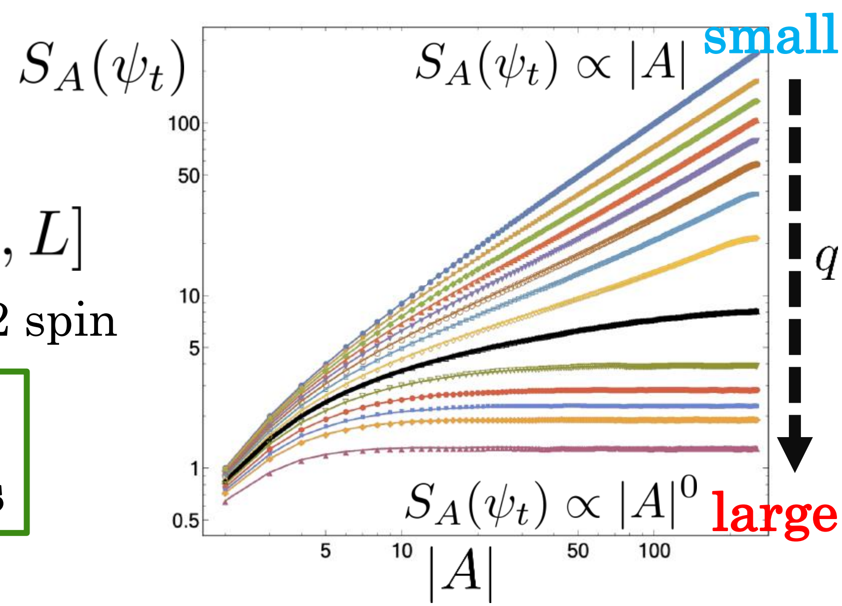
Measurement-induced entanglement transition



$S_A(\psi_t) = -\mathrm{tr}_A\left(\rho_A\left(\psi_t\right)\log\left[\rho_A\left(\psi_t\right)\right]\right)$
an indicator of how complex the state is

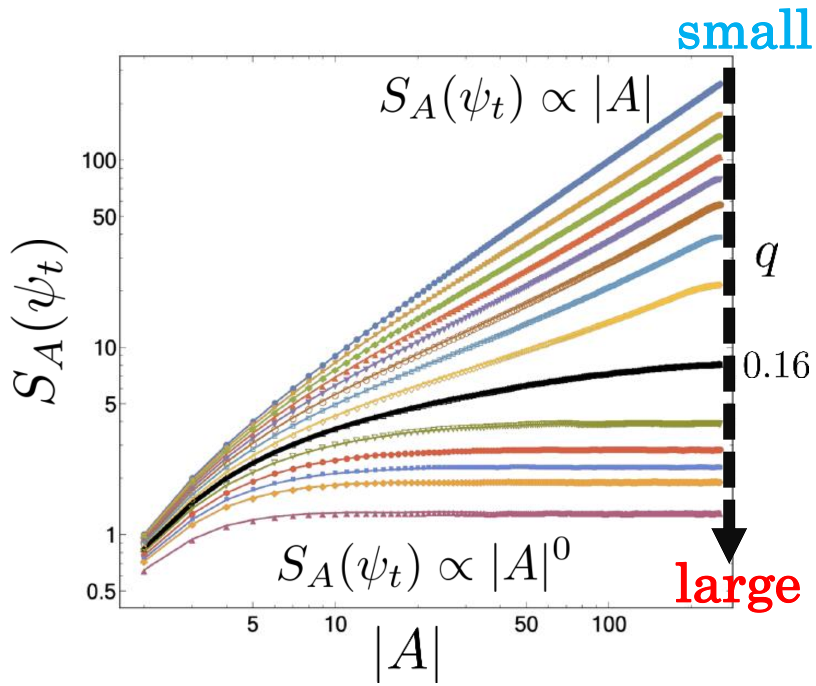


q :probability of measurement



Y. Li et.al. PRB **100**, 134306 (2019).

Mutual information

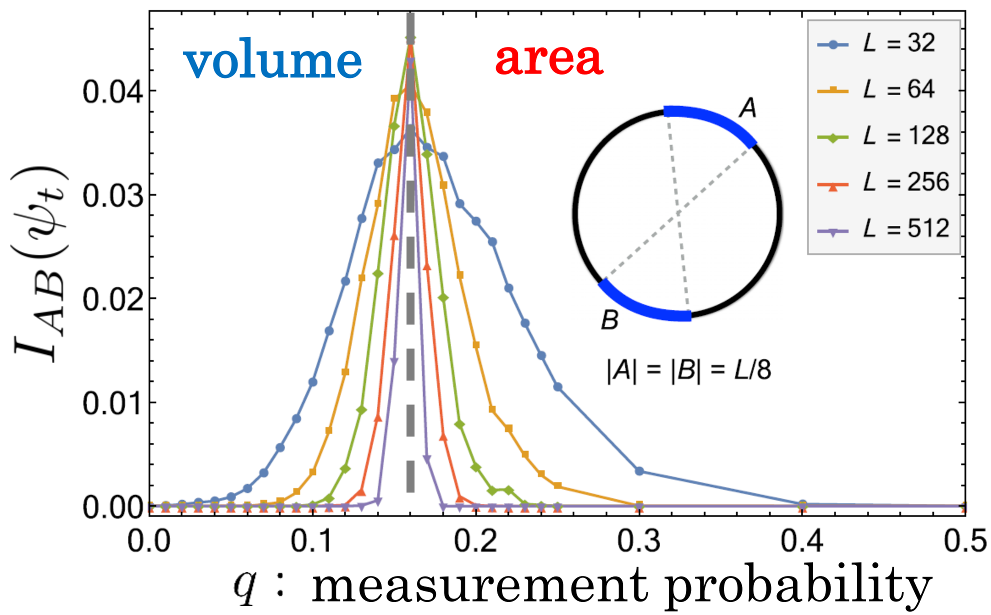


Y. Li et.al., PRB **100**, 134306 (2019).

M. M. Wolf et.al., PRL **100**, 070502 (2008).

$$I_{AB}(\psi) = S_A(\psi) + S_B(\psi) - S_{AB}(\psi)$$

$$\frac{|\langle O_A O_B\rangle_\psi - \langle O_A\rangle_\psi \langle O_B\rangle_\psi|^2}{2|O_A|^2|O_B|^2} \leq I_{AB}(\psi)$$

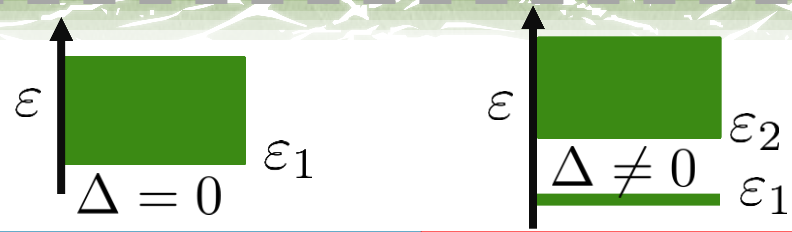


entanglement transition ⟺ peak of $I_{AB}(\psi_t)$

Comparison of isolated and monitored systems

isolated systems:

$$H\left|\Psi_i\right\rangle = \varepsilon_i \left|\Psi_i\right\rangle$$



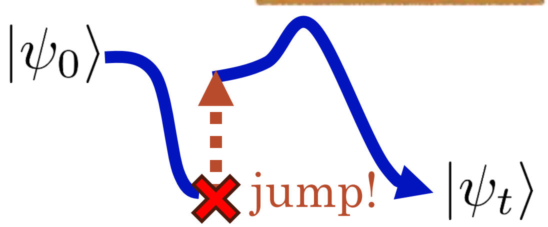
log-law phase gapless phase	area-law phase gapped phase
$S_A(\Psi_1) \propto \log(\lvert A\rvert)$	$S_A(\Psi_1) \propto \lvert A\rvert^0$

→ parameter

monitored systems:

volume-law phase	area-law phase
$S_A(\psi_t) \propto \lvert A\rvert$	$S_A(\psi_t) \propto \lvert A\rvert^0$

→ parameter



Entanglement transition ⟷ Spectral feature ???
🚫 Conventional spectral analysis seems difficult

temporal randomness due to measurement ✖

➡ **no static generator** $\left|\psi_t\right\rangle \neq e^{-iHt}\left|\psi_0\right\rangle$

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Objective, results

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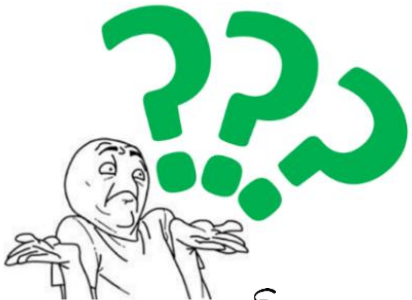
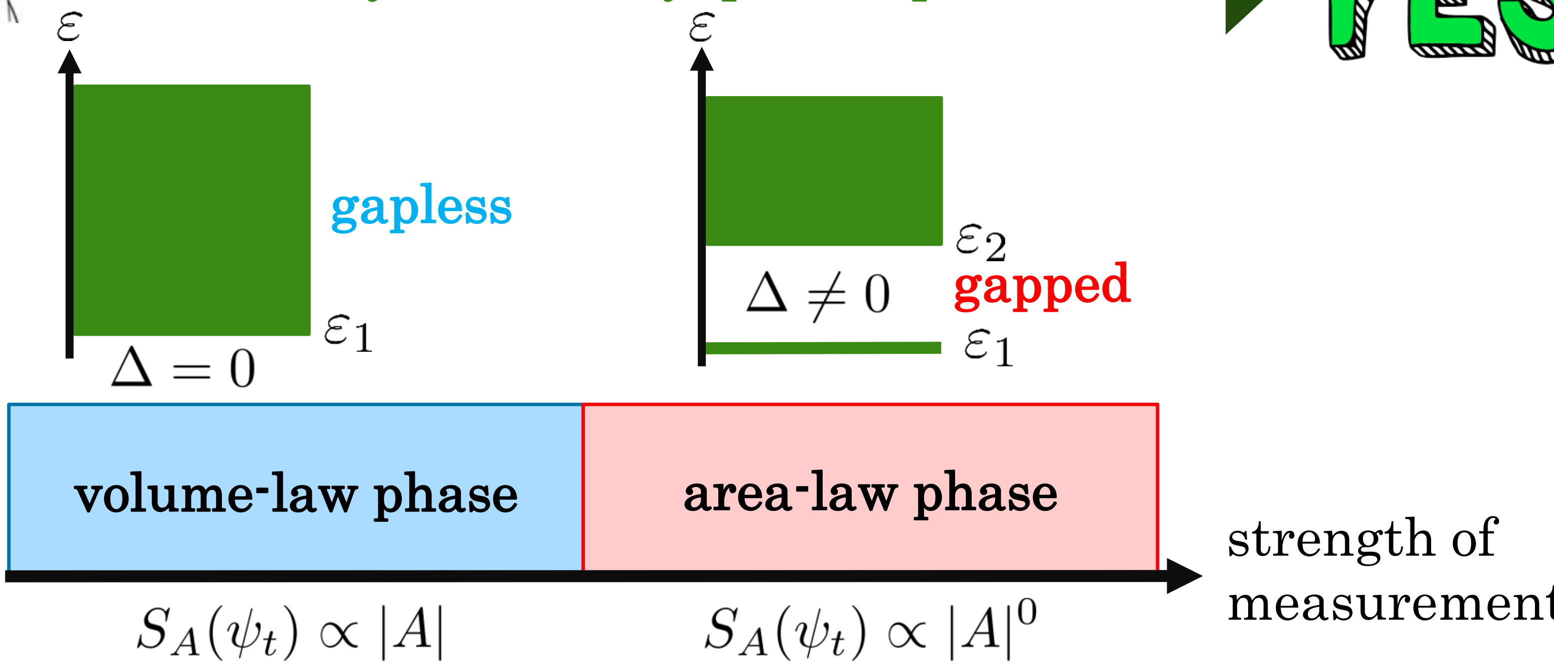


Measurement-induced entanglement transitions are related to some spectral features??

analysis of the Lyapunov spectrum → **YES**



quantum measurements
⇒ entanglement transitions without static generator $|\psi_t\rangle \neq e^{-iHt}|\psi_0\rangle$

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Model

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random nonunitary dynamics

$$\boxed{|\psi_{\boldsymbol{\omega}_t}(\eta)\rangle \propto Q_{\omega_t}(\eta)\cdots Q_{\omega_1}(\eta)\,|\psi_0\rangle}$$

$$Q_{\omega_t}(\eta) = M_{\omega_t}(\eta)U_t$$

$$\boldsymbol{\omega}_t = (\omega_t, \omega_{t-1}, \cdots, \omega_1)$$

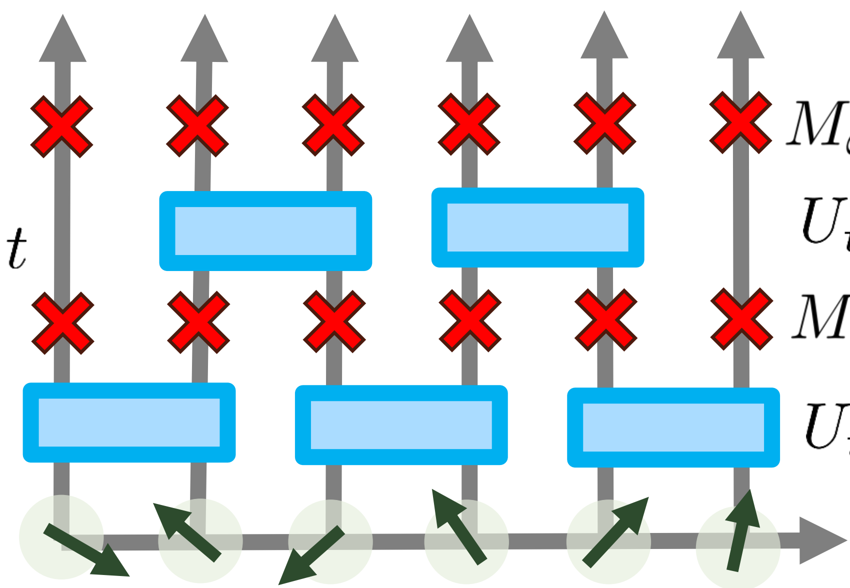
Born probability:

$$P_{\psi_{\boldsymbol{\omega}_{t-1}}(\eta)}(\omega_t) = \left|Q_{\omega_t}(\eta)\,|\psi_{\boldsymbol{\omega}_{t-1}}(\eta)\rangle\right|^2$$

✖ generalized measurement

$$\mathsf{M}_{\omega_{t,x}=\pm}(\eta) = \frac{\sigma_x^0 \pm \eta\sigma_x^3}{\sqrt{2(1+\eta^2)}}$$

η : strength of measurement



$M_{\omega_{t=2}}(\eta)$

$U_{t=2}$

$M_{\omega_{t=1}}(\eta)$

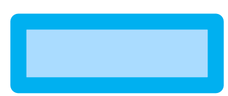
$U_{t=1}$

$x = 1, 2, \cdots, L$

$\omega_{t,x} = \pm$: measurement outcomes

$$M_{\omega_t}(\eta) = \prod_{x=1}^{L} \mathsf{M}_{\omega_{t,x}}(\eta)$$

$$\omega_t = (\omega_{t,x=1}, \cdots, \omega_{t,x=L})$$

 unitary gate

 1/2 spin

Lyapunov analysis

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$$|\psi_{\boldsymbol{\omega}_t}(\eta)\rangle \propto V_{\boldsymbol{\omega}_t}(\eta)\,|\psi_0\rangle\,, \quad V_{\boldsymbol{\omega}_t}(\eta) = Q_{\omega_t}(\eta)Q_{\omega_{t-1}}(\eta)\cdots Q_{\omega_1}(\eta)$$

effective "Hamiltonian": $H_{\boldsymbol{\omega}_t}(\eta) = -\dfrac{1}{2t}\log\left[V_{\boldsymbol{\omega}_t}(\eta)V^\dagger_{\boldsymbol{\omega}_t}(\eta)\right]$

intuitive picture

imaginary-time evolution: $|\psi_{\boldsymbol{\omega}_t}(\eta)\rangle \sim \exp[-H_{\boldsymbol{\omega}_t}(\eta)t]\,|\psi_0\rangle$

$$H_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle = \varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle\,, \quad \varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta) \le \varepsilon^{i+1,L}_{\boldsymbol{\omega}_t}(\eta)$$

$$\Lambda^{i,L}_{\boldsymbol{\omega}_t}(\eta) = \exp\left[-\varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta)t\right]$$

singular-value decomposition:

$$V_{\boldsymbol{\omega}_t}(\eta) = \sum_i \Lambda^{i,L}_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle\,\langle\Phi^i_{\boldsymbol{\omega}_t}(\eta)| \simeq |\Psi^1_{\boldsymbol{\omega}_t}(\eta)\rangle\,\langle\Phi^1_{\boldsymbol{\omega}_t}(\eta)|$$

$$\Rightarrow\ |\psi_{\boldsymbol{\omega}_t}(\eta)\rangle \simeq |\Psi^1_{\boldsymbol{\omega}_t}(\eta)\rangle \quad \left[\Lambda^{1,L}_{\boldsymbol{\omega}_t}(\eta) \gg \Lambda^{2,L}_{\boldsymbol{\omega}_t}(\eta)\right]$$

$$V_{\boldsymbol{\omega}_t}(\eta)V^\dagger_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle = \Lambda^{i,L}_{\boldsymbol{\omega}_t}(\eta)^2\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle$$

$$V^\dagger_{\boldsymbol{\omega}_t}(\eta)V_{\boldsymbol{\omega}_t}(\eta)\,|\Phi^i_{\boldsymbol{\omega}_t}(\eta)\rangle = \Lambda^{i,L}_{\boldsymbol{\omega}_t}(\eta)^2\,|\Phi^i_{\boldsymbol{\omega}_t}(\eta)\rangle$$

Lyapunov analysis

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PRL **134**, 010410 (2025).

$$|\psi_{\boldsymbol{\omega}_t}(\eta)\rangle \propto V_{\boldsymbol{\omega}_t}(\eta)\,|\psi_0\rangle\,,\quad V_{\boldsymbol{\omega}_t}(\eta) = Q_{\omega_t}(\eta)Q_{\omega_{t-1}}(\eta)\cdots Q_{\omega_1}(\eta)$$

effective "Hamiltonian": $H_{\boldsymbol{\omega}_t}(\eta) = -\dfrac{1}{2t}\log\left[V_{\boldsymbol{\omega}_t}(\eta)V^{\dagger}_{\boldsymbol{\omega}_t}(\eta)\right]$

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$$H_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^{i}_{\boldsymbol{\omega}_t}(\eta)\rangle = \varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^{i}_{\boldsymbol{\omega}_t}(\eta)\rangle\,,\quad \varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta) \leq \varepsilon^{i+1,L}_{\boldsymbol{\omega}_t}(\eta)$$

spectral gap: $\Delta_L(\eta) = \lim_{t\to\infty}\Delta^{L}_{\boldsymbol{\omega}_t}(\eta),\ \Delta^{L}_{\boldsymbol{\omega}_t}(\eta) = \varepsilon^{2,L}_{\boldsymbol{\omega}_t}(\eta) - \varepsilon^{1,L}_{\boldsymbol{\omega}_t}(\eta)$

ground state of the effective Hamiltonian : $|\Psi^{1}_{\boldsymbol{\omega}_t}(\eta)\rangle$

analogy to ground-state transitions in isolated quantum systems

relaxation time:

$$t \gg 1/\Delta_L(\eta) \to |\psi_{\boldsymbol{\omega}_t}(\eta)\rangle \simeq |\Psi^{1}_{\boldsymbol{\omega}_t}(\eta)\rangle$$

Spectral gap

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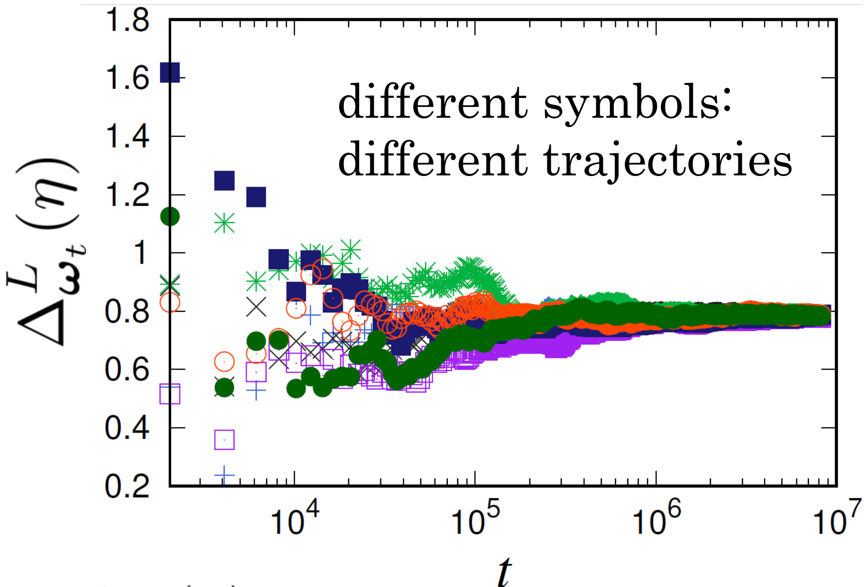
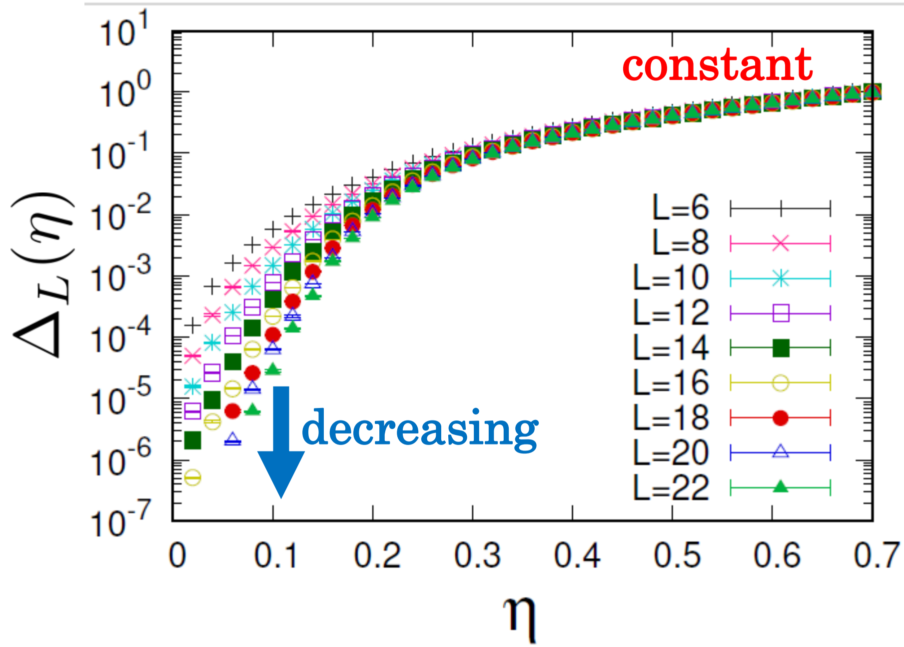
$$H_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle = \varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle\,, \quad \varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta) \le \varepsilon^{i+1,L}_{\boldsymbol{\omega}_t}(\eta)$$

spectral gap: $\Delta_L(\eta) = \lim_{t\to\infty} \Delta^L_{\boldsymbol{\omega}_t}(\eta),\ \ \Delta^L_{\boldsymbol{\omega}_t}(\eta) = \varepsilon^{2,L}_{\boldsymbol{\omega}_t}(\eta) - \varepsilon^{1,L}_{\boldsymbol{\omega}_t}(\eta)$





$\Delta_L(\eta)$: independent of trajectories

global quantity!

η : measurement strength

L : system size

small η : $\lim_{L\to\infty} \Delta_L(\eta) = 0$

large η : $\lim_{L\to\infty} \Delta_L(\eta) \neq 0$

Spectral transition

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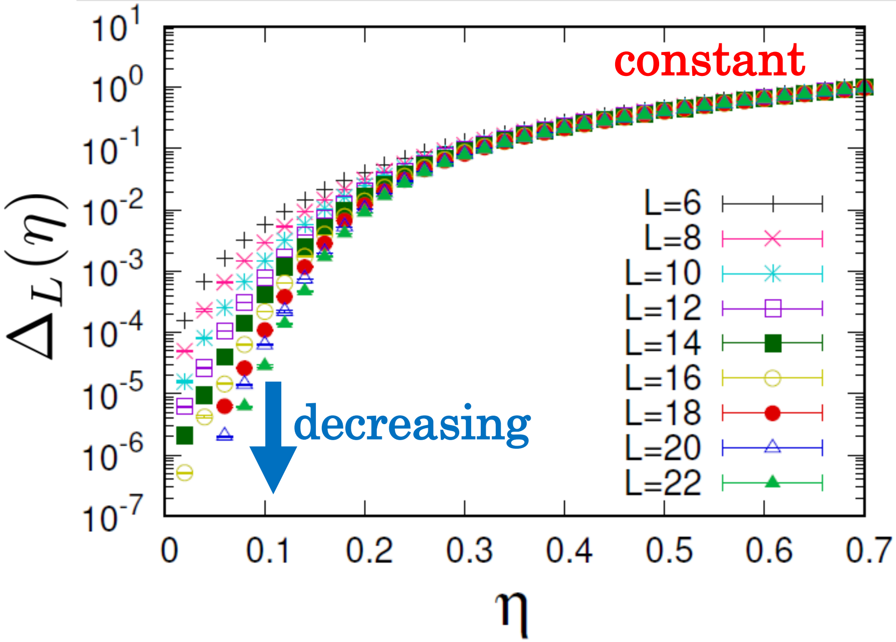
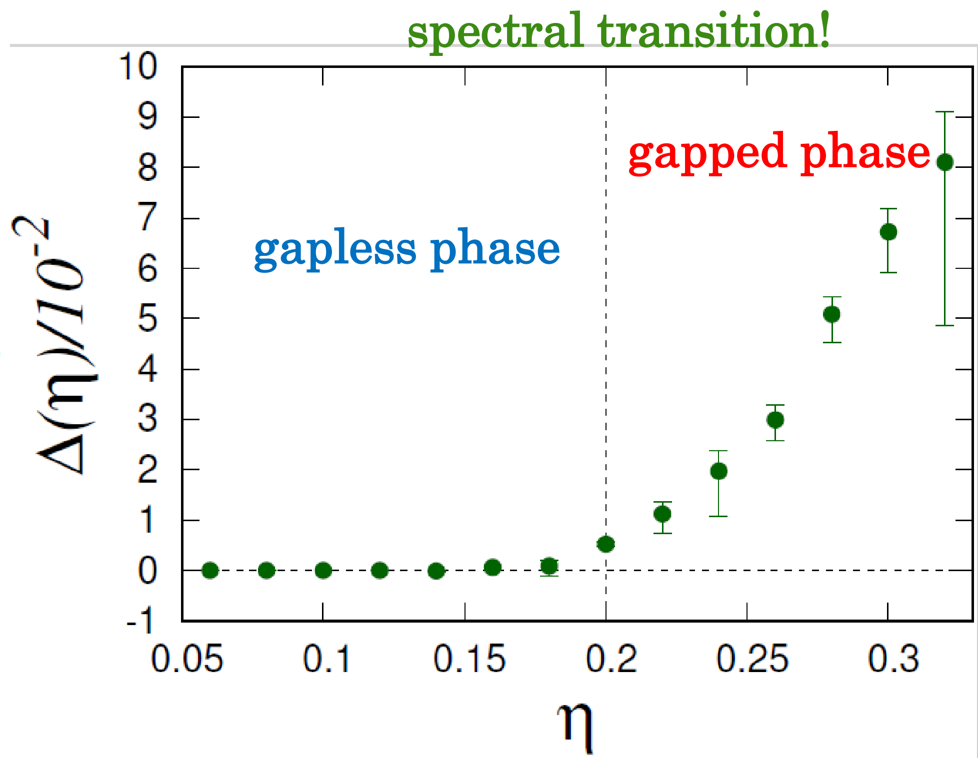
spectral gap: $\Delta_L(\eta) = \lim_{t\to\infty} \Delta^{L}_{\boldsymbol{\omega}_t}(\eta), \ \ \Delta^{L}_{\boldsymbol{\omega}_t}(\eta) = \varepsilon^{2,L}_{\boldsymbol{\omega}_t}(\eta) - \varepsilon^{1,L}_{\boldsymbol{\omega}_t}(\eta)$



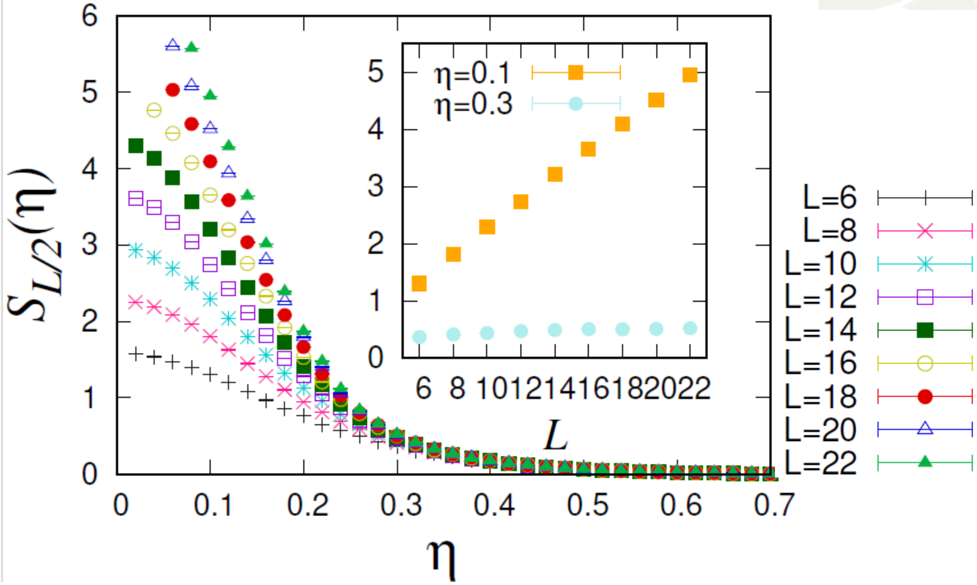


fitting:

$$\Delta^{\rm fit}_L(\eta) = \Delta(\eta) + \alpha(\eta)[\beta(\eta)]^{-L}$$

$$\Delta(\eta) = \lim_{L\to\infty} \Delta^{\rm fit}_L(\eta)$$

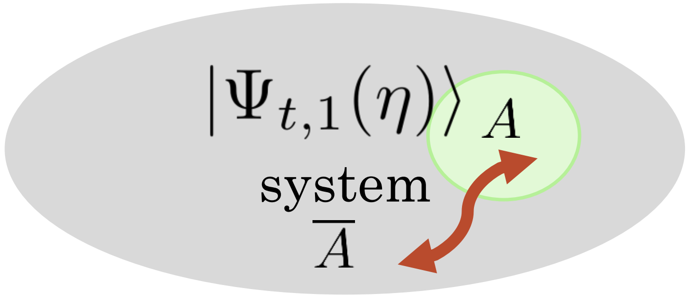
Ground-state entanglement entropy



$$S_{L/2}(\eta) = \mathbb{E}_t\left(S_{L/2}\left[\Psi^1_{\boldsymbol{\omega}_t}(\eta)\right]\right)$$

$$S_A(\psi) = -\mathrm{tr}_A\left(\rho_A(\psi)\log\left[\rho_A(\psi)\right]\right)$$

$\mathbb{E}_t$:time average $\quad \rho_A(\psi) = \mathrm{tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$



small η : volume law $S_{L/2}(\eta) \propto L$

large η : area law $S_{L/2}(\eta) \propto L^0$

large $S_A(\psi)$

$\Rightarrow A$ and $\overline{A}$ are highly entangled

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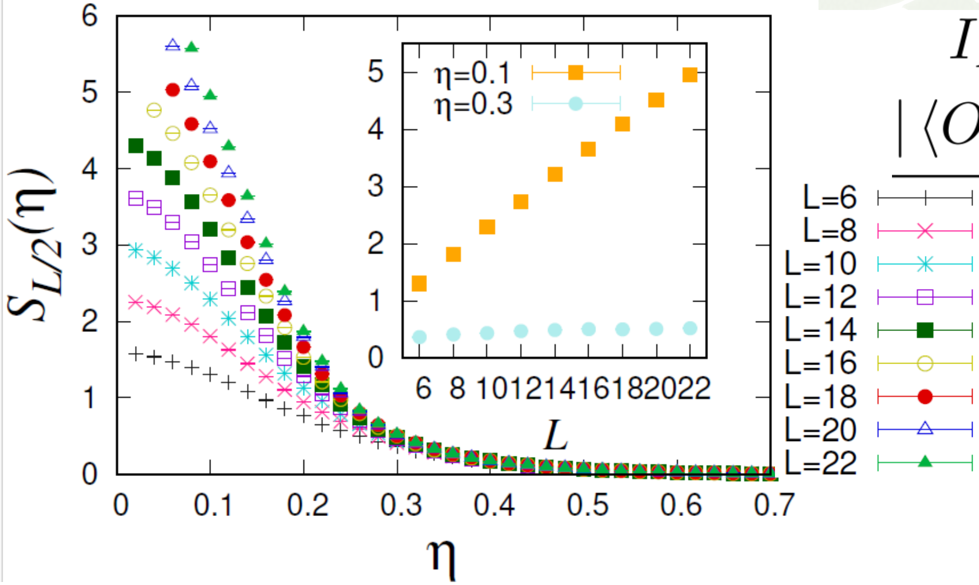
Ground-state entanglement entropy



$$I_{AB}(\psi) = S_A(\psi) + S_B(\psi) - S_{AB}(\psi)$$

$$\frac{|\langle O_A O_B\rangle_\psi - \langle O_A\rangle_\psi \langle O_B\rangle_\psi|^2}{2|O_A|^2|O_B|^2} \leq I_{AB}(\psi)$$

M. M. Wolf et.al., PRL **100**, 070502 (2008).

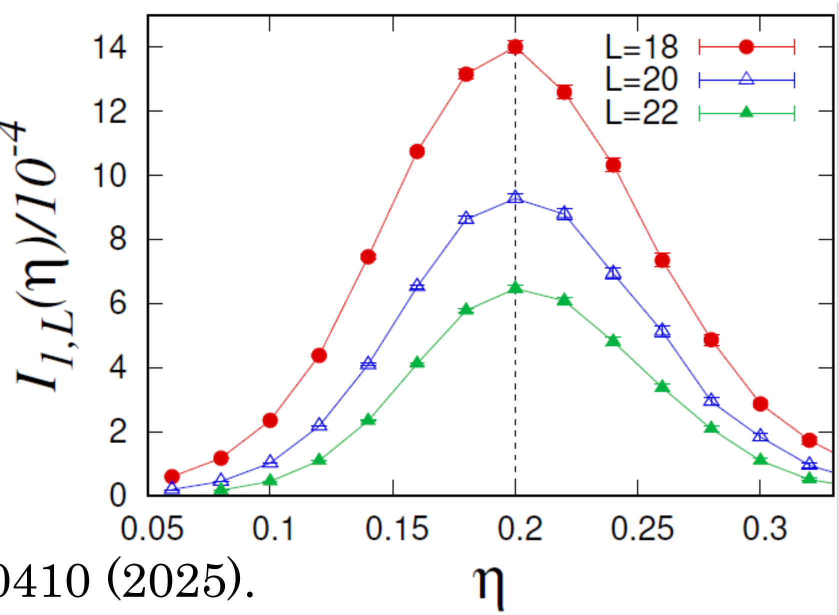
peak of $I_{1,L}(\eta) = \mathbb{E}_t\left(I_{1,L}\left[\Psi^1_{\boldsymbol{\omega}_t}(\eta)\right]\right)$

phase transition!

$$S_{L/2}(\eta) = \mathbb{E}_t\left(S_{L/2}\left[\Psi^1_{\boldsymbol{\omega}_t}(\eta)\right]\right)$$

$$S_A(\psi) = -\mathrm{tr}_A\left(\rho_A(\psi)\log\left[\rho_A(\psi)\right]\right)$$

$\mathbb{E}_t$:time average $\quad \rho_A(\psi) = \mathrm{tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$



small η : volume law $S_{L/2}(\eta) \propto L$

large η : area law $S_{L/2}(\eta) \propto L^0$

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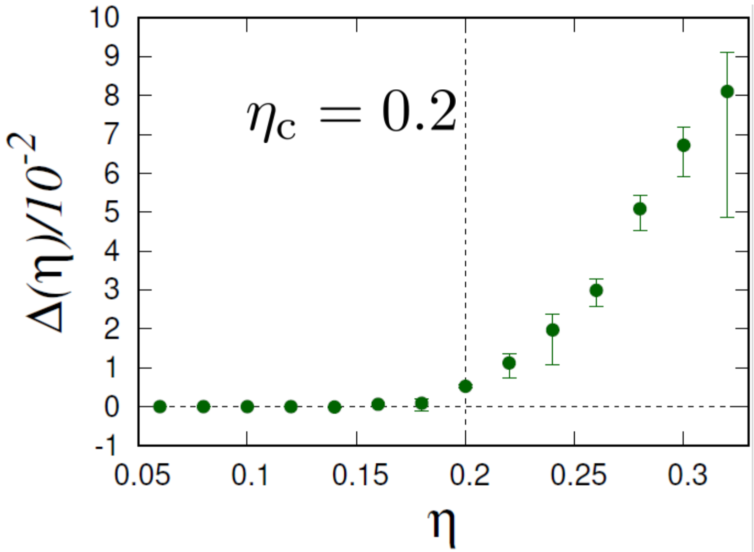
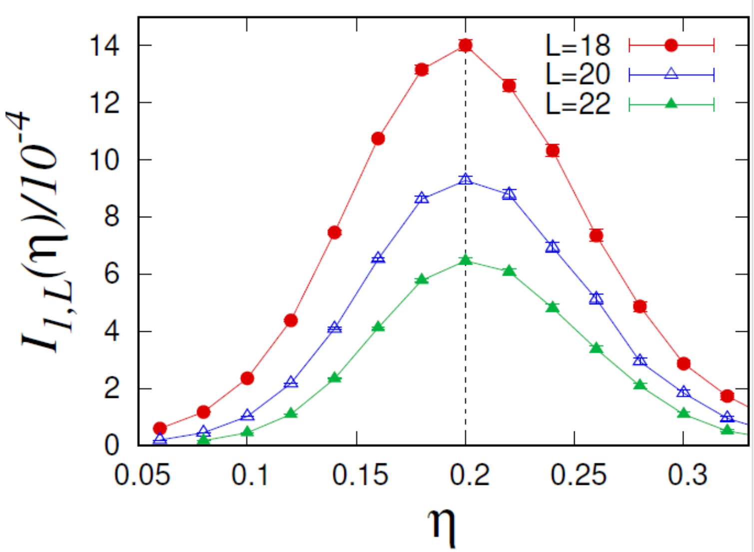
Coincidence of the thresholds





$$H_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle = \varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle$$

gapless phase = volume-law phase

$\Delta(\eta) = 0 \qquad S_{L/2}(\eta) \propto L$

gapped phase = area-law phase

$\Delta(\eta) \neq 0 \qquad S_{L/2}(\eta) \propto L^0$

spectral transition ⇔ entanglement transition: ubiquitous in a wide range of quantum systems

Ken Mochizuki, Ryusuke Hamazaki,
Physical Review Letters **134**, 010410 (2025).

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Typical convergence of the Lyapunov spectrum

$$V_{\boldsymbol{\omega}_t}(\eta)V^{\dagger}_{\boldsymbol{\omega}_t}(\eta)\left|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\right\rangle = e^{-\varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta)t}\left|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\right\rangle$$

$$\boxed{\varepsilon_{i,L}(\eta) = \lim_{t\to\infty}\varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta)}$$

why???

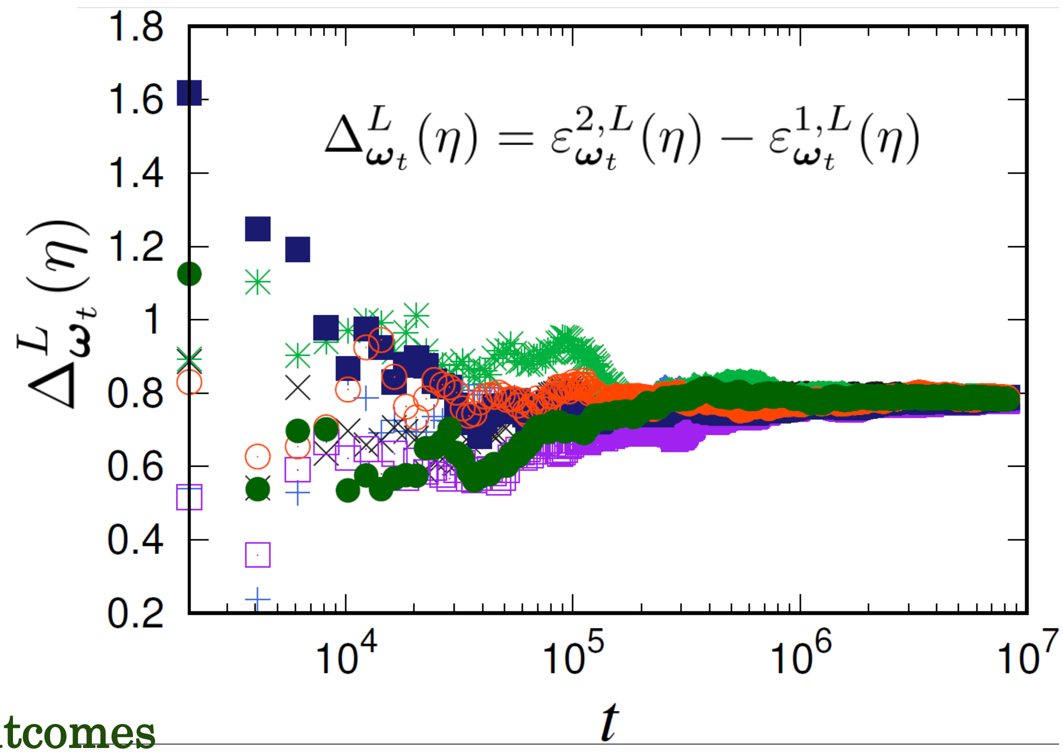
$$V_{\boldsymbol{\omega}_t}(\eta) = Q_{\omega_t}(\eta)\cdots Q_{\omega_1}(\eta)$$

$$\left|\psi_{\boldsymbol{\omega}_t}(\eta)\right\rangle \propto V_{\boldsymbol{\omega}_t}(\eta)\left|\psi_0\right\rangle$$

$$P_{\psi_{\boldsymbol{\omega}_{t-1}}(\eta)}(\omega_t) = \left|Q_{\omega_t}(\eta)\left|\psi_{\boldsymbol{\omega}_{t-1}}(\eta)\right\rangle\right|^2$$

products of random matrices obeying the Born rule

spectrum typically independent of initial states and measurement outcomes



In the case of i.i.d outcomes

Ludwig Arnold, Random Dynamical Systems, Springer

products of random non-unitary matrices:

$$V(\boldsymbol{\omega}_t) = Q(\omega_t)Q(\omega_{t-1})\cdots Q(\omega_1)$$

random outcomes:

$$\boldsymbol{\omega}_t = (\omega_1, \omega_2, \cdots, \omega_t)$$

$\{\omega_t\}$ are independently and identically distributed

$\Rightarrow$ the probability of $\boldsymbol{\omega}_t$: $P(\boldsymbol{\omega}_t) = P(\omega_1)P(\omega_2)\cdots P(\omega_t)$

Ω_t : a set of $\boldsymbol{\omega}_t$

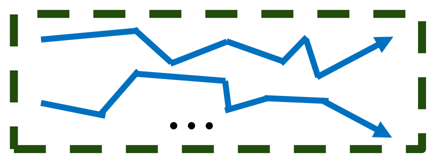
shift of time: $\theta\boldsymbol{\omega}_t = (\omega_2, \omega_3, \cdots)$

e.g. $\Omega_\infty = \{\boldsymbol{\omega}_\infty | \omega_t = \uparrow \ \forall t\}$ (one spin) $\Rightarrow P(\Omega_\infty) = P^\infty(\uparrow) = 0$: neglected

e.g. $\Omega_\infty = \{\boldsymbol{\omega}_\infty | \omega_3 = \downarrow \text{ and } \omega_8 = \downarrow\}$ (one spin) $\Rightarrow P(\Omega_\infty) = P^2(\downarrow) \neq 0$

goal: $\lim_{t\to\infty} -\frac{1}{t}\log\left[\Lambda_i(\boldsymbol{\omega}_t)\right] = \varepsilon_i$

typically independent of $\boldsymbol{\omega}_\infty$



$P(\Omega_t) = 0$

atypical, neglected

ε_i dependent on $\boldsymbol{\omega}_\infty$

*Kingman's subadditive ergodic theorem

$$V(\boldsymbol{\omega}_t)V^\dagger(\boldsymbol{\omega}_t)\left|\Psi_i(\boldsymbol{\omega}_t)\right\rangle = \Lambda_i^2(\boldsymbol{\omega}_t)\left|\Psi_i(\boldsymbol{\omega}_t)\right\rangle$$

In the case of i.i.d outcomes

Ludwig Arnold, Random Dynamical Systems, Springer

products of random non-unitary matrices:

$$V(\boldsymbol{\omega}_t) = Q(\omega_t)Q(\omega_{t-1})\cdots Q(\omega_1)$$

random outcomes:

$$\boldsymbol{\omega}_t = (\omega_1, \omega_2, \cdots, \omega_t)$$

$\{\omega_t\}$ are independently and identically distributed

$\Rightarrow$ the probability of $\boldsymbol{\omega}_t$: $P(\boldsymbol{\omega}_t) = P(\omega_1)P(\omega_2)\cdots P(\omega_t)$

Ω_t : a set of $\boldsymbol{\omega}_t$

shift of time: $\theta\boldsymbol{\omega}_t = (\omega_2, \omega_3, \cdots)$

- invariant measure $P(\boldsymbol{\omega}_t \in \Omega_t) = P \circ \theta^{-1}(\Omega_t) = P(\theta\boldsymbol{\omega}_t \in \Omega_t)$
- ergodicity $\Omega_t = \theta^{-1}(\Omega_t) \Rightarrow P(\Omega_t) = 0, 1$ (roughly speaking)

e.g. $\Omega_t = \{\boldsymbol{\omega}_t | \omega_1 = \uparrow \text{ and } \omega_t = \downarrow\} \Rightarrow P(\Omega_t) = P \circ \theta^{-1}(\Omega_t) = P(\uparrow)P(\downarrow)$

goal: $\displaystyle\lim_{t\to\infty} -\frac{1}{t}\log\left[\Lambda_i(\boldsymbol{\omega}_t)\right] = \varepsilon_i$

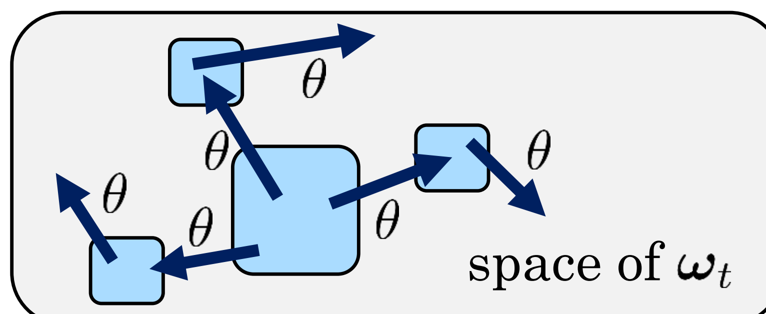
typically independent of $\boldsymbol{\omega}_\infty$

*Kingman's subadditive ergodic theorem

$$V(\boldsymbol{\omega}_t)V^\dagger(\boldsymbol{\omega}_t)\left|\Psi_i(\boldsymbol{\omega}_t)\right\rangle = \Lambda_i^2(\boldsymbol{\omega}_t)\left|\Psi_i(\boldsymbol{\omega}_t)\right\rangle$$

picture of ergodicity:

θ covers the whole space of $\boldsymbol{\omega}_t$



In the case of quantum measurements

$$V(\boldsymbol{\omega}_t) = Q(\omega_t)Q(\omega_{t-1})\cdots Q(\omega_1)$$

$$\boldsymbol{\omega}_t = (\omega_1, \omega_2, \cdots, \omega_t)$$

$$\theta\boldsymbol{\omega}_t = (\omega_2, \omega_3, \cdots)$$

If $\{\omega_t\}$ obey the Born rule $\underline{P_\rho(\boldsymbol{\omega}_t) = \mathrm{tr}\left[V(\boldsymbol{\omega}_t)\rho V^\dagger(\boldsymbol{\omega}_t)\right]}$, $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$

$$\cancel{P(\boldsymbol{\omega}_t) = P(\omega_1)P(\omega_2)\cdots P(\omega_t)}$$

Ω_t : a set of $\boldsymbol{\omega}_t$

- invariant measure $P(\boldsymbol{\omega}_t \in \Omega_t) = P \circ \theta^{-1}(\Omega_t) = P(\theta\boldsymbol{\omega}_t \in \Omega_t)$
- ergodicity $\Omega_t = \theta^{-1}(\Omega_t) \Rightarrow P(\Omega_t) = 0, 1$ (roughly speaking)

goal: $\lim_{t\to\infty} -\frac{1}{t}\log\left[\Lambda_i(\boldsymbol{\omega}_t)\right] = \varepsilon_i$

typically independent of ω_∞

? ? ?

? ? ?

$$V(\boldsymbol{\omega}_t)V^\dagger(\boldsymbol{\omega}_t)|\Psi_i(\boldsymbol{\omega}_t)\rangle = \Lambda_i^2(\boldsymbol{\omega}_t)|\Psi_i(\boldsymbol{\omega}_t)\rangle$$

Condition for typical convergence

averaged dynamics:

$$\rho_{\tau+1} = \Gamma(\rho_\tau) = \sum_\omega Q(\omega)\rho_\tau Q^\dagger(\omega)$$

time-dependent measure:

$$P_{\rho_\tau}(\boldsymbol{\omega}_t) = \mathrm{tr}\left[V(\boldsymbol{\omega}_t)\rho_\tau V^\dagger(\boldsymbol{\omega}_t)\right]$$

$$P_{\rho_\tau}(\theta\boldsymbol{\omega}_t \in \Omega_t) = P_{\rho_{\tau+1}}(\boldsymbol{\omega}_t \in \Omega_t)$$

$$V(\boldsymbol{\omega}_t) = Q(\omega_t)\cdots Q(\omega_1)$$

T. Benoist et.al., Probability Theory and Related Fields **174**, 307 (2019).

- invariant measure
- ergodicity (roughly speaking)

$$\lim_{t\to\infty} -\frac{1}{t}\log\left[\Lambda_i(\boldsymbol{\omega}_t)\right] = \varepsilon_i$$

typically independent of ω_∞

? ? ?

? ? ?

$$\rho = \sum_i p_i \left|\phi_i\right\rangle\left\langle\phi_i\right|$$

Condition for typical convergence

averaged dynamics:

$$\rho_{\tau+1} = \Gamma(\rho_\tau) = \sum_\omega Q(\omega)\rho_\tau Q^\dagger(\omega)$$

time-dependent measure:

$$P_{\rho_\tau}(\boldsymbol{\omega}_t) = \mathrm{tr}\left[V(\boldsymbol{\omega}_t)\rho_\tau V^\dagger(\boldsymbol{\omega}_t)\right]$$

$$P_{\rho_\tau}(\theta\boldsymbol{\omega}_t \in \Omega_t) = P_{\rho_{\tau+1}}(\boldsymbol{\omega}_t \in \Omega_t)$$

$$V(\boldsymbol{\omega}_t) = Q(\omega_t)\cdots Q(\omega_1)$$

● **unique and positive-definite stationary state** $\rho_\infty = \Gamma(\rho_\infty) > 0$ **(present in our model)**

T. Benoist et.al., Probability Theory and Related Fields **174**, 307 (2019).

■ invariant measure $P_{\rho_\infty}(\boldsymbol{\omega}_t \in \Omega_t) = P_{\rho_\infty}(\theta\boldsymbol{\omega}_t \in \Omega_t)$

■ ergodicity $\Omega_t = \theta^{-1}(\Omega_t) \Rightarrow P_{\rho_\infty}(\Omega_t) = 0, 1$ (roughly speaking)

$$\lim_{t\to\infty} -\frac{1}{t}\log\left[\Lambda_i(\boldsymbol{\omega}_t)\right] = \varepsilon_i$$

typically independent of $\boldsymbol{\omega}_\infty$

$$\rho = \sum_i p_i \left|\phi_i\right\rangle\left\langle\phi_i\right|$$

Ergodicity

● unique, positive-definite stationary state $\rho_\infty = \Gamma(\rho_\infty)$

■ ergodicity $\Omega_t = \theta^{-1}(\Omega_t) \Rightarrow P_{\rho_\infty}(\Omega_t) = 0, 1$

$$\rho_{\tau+1} = \Gamma(\rho_\tau)$$

$$P_{\rho_\infty}\left[(\boldsymbol{\omega}_t \in \Omega_t) \cap \left(\theta^T \boldsymbol{\omega}'_t \in \Omega'_t\right)\right] \qquad \boldsymbol{\omega}_\infty = (\boldsymbol{\omega}_t, \cdots, \boldsymbol{\omega}'_t, \cdots)$$

$$= \sum_{\boldsymbol{\omega}_t \in \Omega_t} \sum_{\boldsymbol{\omega}'_t \in \Omega'_t} \mathrm{tr}\left[V(\boldsymbol{\omega}'_t)\Gamma^{T-t}[V(\boldsymbol{\omega}_t)\rho_\infty V^\dagger(\boldsymbol{\omega}_t)]V^\dagger(\boldsymbol{\omega}'_t)\right]$$

$$\bigstar \simeq \sum_{\boldsymbol{\omega}_t \in \Omega_t} \mathrm{tr}\left[V(\boldsymbol{\omega}_t)\rho_\infty V^\dagger(\boldsymbol{\omega}_t)\right] \sum_{\boldsymbol{\omega}'_t \in \Omega'_t} \mathrm{tr}\left[V(\boldsymbol{\omega}'_t)\rho_\infty V^\dagger(\boldsymbol{\omega}'_t)\right]$$

$$= P_{\rho_\infty}(\Omega_t)\, P_{\rho_\infty}(\Omega'_t) \quad \text{for large } T$$

$$\bigstar\ \Gamma^{T-t}[V(\boldsymbol{\omega}_t)\rho_\infty V^\dagger(\boldsymbol{\omega}_t)] \simeq \mathrm{tr}\left[V(\boldsymbol{\omega}_t)\rho_\infty V^\dagger(\boldsymbol{\omega}_t)\right]\rho_\infty$$

$$\therefore\ \Omega'_t = \Omega_t \text{ and } \Omega_t = \theta^{-1}(\Omega_t) \Rightarrow \underline{P_{\rho_\infty}(\Omega_t) = P^2_{\rho_\infty}(\Omega_t)}$$

T. Benoist et.al., Probability Theory and Related Fields **174**, 307 (2019).

Typical convergence in arbitrary initial states

● unique and positive-definite stationary state $\rho_\infty = \Gamma(\rho_\infty) > 0$ (present in our model)

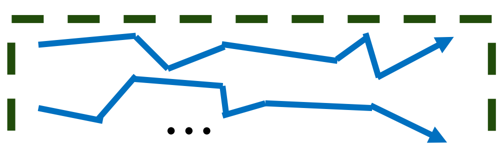
T. Benoist et.al., Probability Theory and Related Fields **174**, 307 (2019).

■ invariant measure $P_{\rho_\infty}(\boldsymbol{\omega}_t \in \Omega_t) = P_{\rho_\infty}(\theta\boldsymbol{\omega}_t \in \Omega_t)$

■ ergodicity $\Omega_t = \theta^{-1}(\Omega_t) \Rightarrow P_{\rho_\infty}(\Omega_t) = 0, 1$ (roughly speaking)

$$\lim_{t\to\infty} -\frac{1}{t}\log\left[\Lambda_i(\boldsymbol{\omega}_t)\right] = \varepsilon_i$$

in typical trajectories from ρ_∞



$P_{\rho_\infty}(\Omega_t) = 0$

atypical, neglected

ε_i dependent on $\boldsymbol{\omega}_\infty$

Typical convergence in arbitrary initial states

● unique and positive-definite stationary state
$\rho_\infty = \Gamma(\rho_\infty) > 0$ (present in our model)

T. Benoist et.al., Probability Theory and Related Fields **174**, 307 (2019).

■ invariant measure $P_{\rho_\infty}(\boldsymbol{\omega}_t \in \Omega_t) = P_{\rho_\infty}(\theta\boldsymbol{\omega}_t \in \Omega_t)$

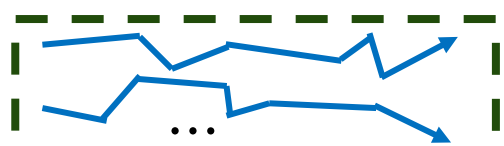
■ ergodicity $\Omega_t = \theta^{-1}(\Omega_t) \Rightarrow P_{\rho_\infty}(\Omega_t) = 0, 1$ (roughly speaking)

$\rho_\infty > 0, \ \ P_\rho(\boldsymbol{\omega}_t) = \mathrm{tr}\left[V(\boldsymbol{\omega}_t)\rho V^\dagger(\boldsymbol{\omega}_t)\right]$

absolute continuity : $P_{\rho_\infty}(\Omega_t) = 0$ leads to $P_\rho(\Omega_t) = 0$ for any ρ

$$\lim_{t\to\infty} -\frac{1}{t}\log\left[\Lambda_i(\boldsymbol{\omega}_t)\right] = \varepsilon_i$$

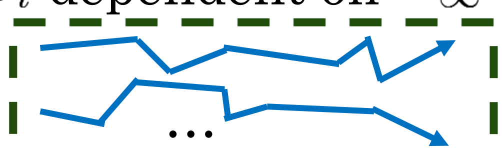
in typical trajectories from any ρ



$P_{\rho_\infty}(\Omega_t) = 0$

atypical, neglected

ε_i dependent on $\boldsymbol{\omega}_\infty$



$P_\rho(\Omega_t) = 0$

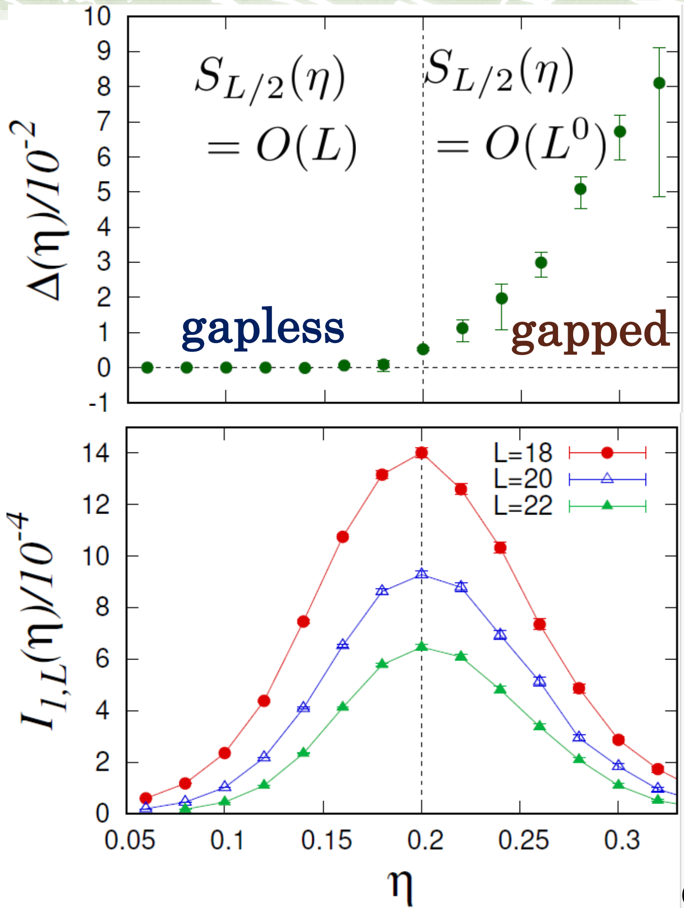
atypical, neglected

Outline

- Introduction:
  - ▶ entanglement, spectrum, phase transitions
  - ▶ isolated systems **without randomness**
  - ▶ **random** dynamics in monitored systems
- Motivation and rough sketch of results
- Our results:
  - ▶ measurement-induced spectral transition
  - ▶ ground-state entanglement transition
- Foundation of the Lyapunov spectral analysis
- Summary

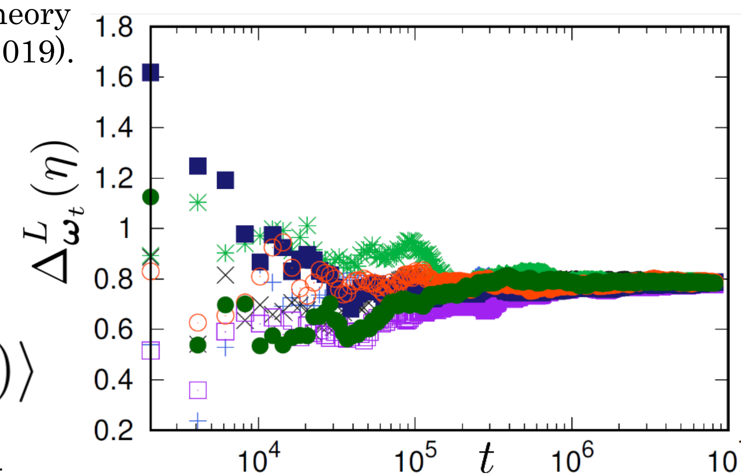
Summary

Ken Mochizuki
Ryusuke Hamazaki
PRL **134**, 010410 (2025).



unique and positive definite stationary state
in averaged dynamics $\rho_\infty = \Gamma(\rho_\infty) > 0$
typical convergence of Lyapunov spectrum
in trajectories $|\psi_{\boldsymbol{\omega}_t}(\eta)\rangle$

T. Benoist et.al., Probability Theory and Related Fields **174**, 307 (2019).



$$H_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle = \varepsilon^{i,L}_{\boldsymbol{\omega}_t}(\eta)\,|\Psi^i_{\boldsymbol{\omega}_t}(\eta)\rangle$$

spectral transition

(analogous to isolated systems)

entanglement transition

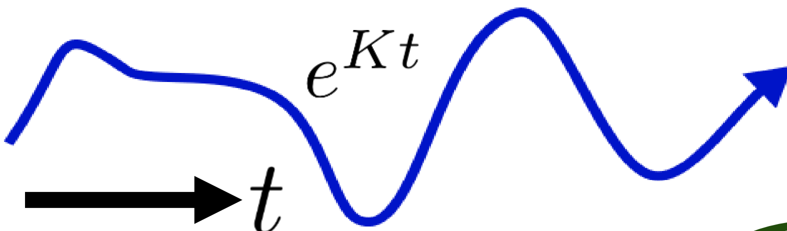
c.f. topological phases and bulk-edge correspondence in monitored dynamics
-Z. Xiao and K. Kawabata, arXiv:2412.06133
-H. Oshima, K. Mochizuki, R. Hamazaki, and Y. Fuji, arXiv:2412.11097

Spectrum in systems with no randomness

systems described by time-independent generators

$$\frac{d}{dt}\left|\psi(t)\right\rangle = -iH\left|\psi(t)\right\rangle, \quad \frac{d}{dt}\hat{\rho}(t) = \mathcal{L}\hat{\rho}(t), \cdots$$

$$\left|\psi(0)\right\rangle, \hat{\rho}(0) \xrightarrow{e^{Kt}} \left|\psi(t)\right\rangle, \hat{\rho}(t)$$



the spectrum of $K = -iH, \mathcal{L}$

↓

essential information of the system

$$\det(\lambda - K) = \cdots$$

Spectral gap in open quantum systems

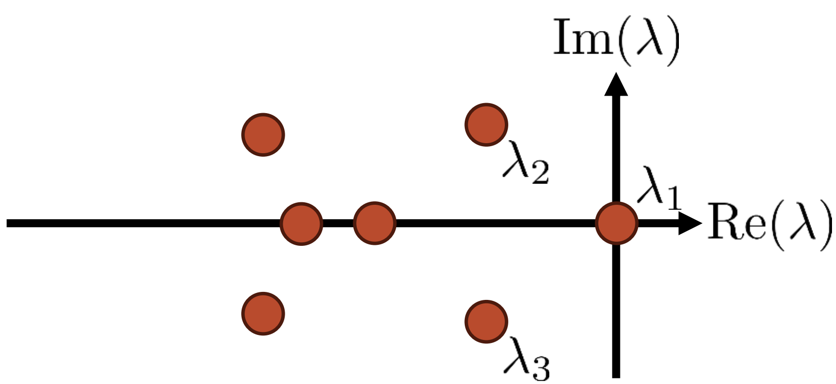
master equation in Markovian open quantum systems:

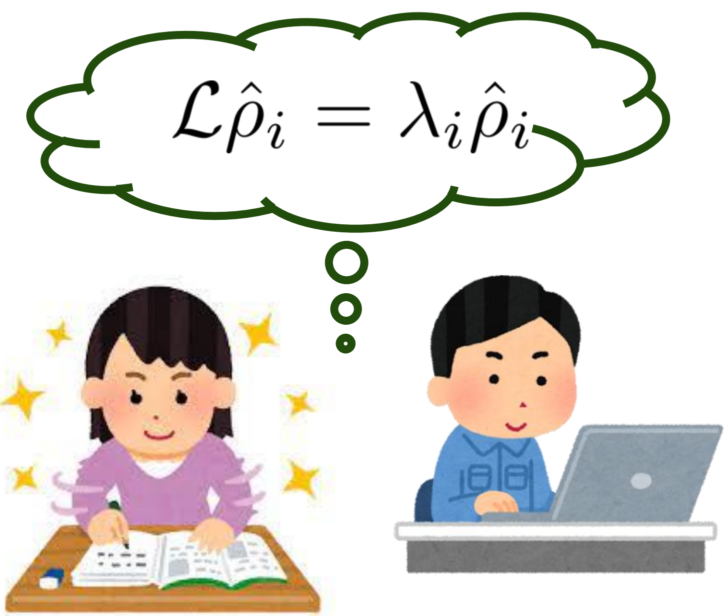
$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}\hat{\rho}(t) \qquad \hat{\rho}(t) = e^{\mathcal{L}t}\hat{\rho}(0) = \sum_i c_i e^{\lambda_i t}\hat{\rho}_i$$

the spectral gap $\Delta = |\mathrm{Re}(\lambda_2) - \mathrm{Re}(\lambda_1)|, \quad \mathrm{Re}(\lambda_j) \geq \mathrm{Re}(\lambda_{j+1})$

→ **the relaxation time to the stationary state** $1/\Delta$

$$\boxed{t \gg \frac{1}{\Delta} \to \hat{\rho}(t) \simeq \hat{\rho}_1}$$



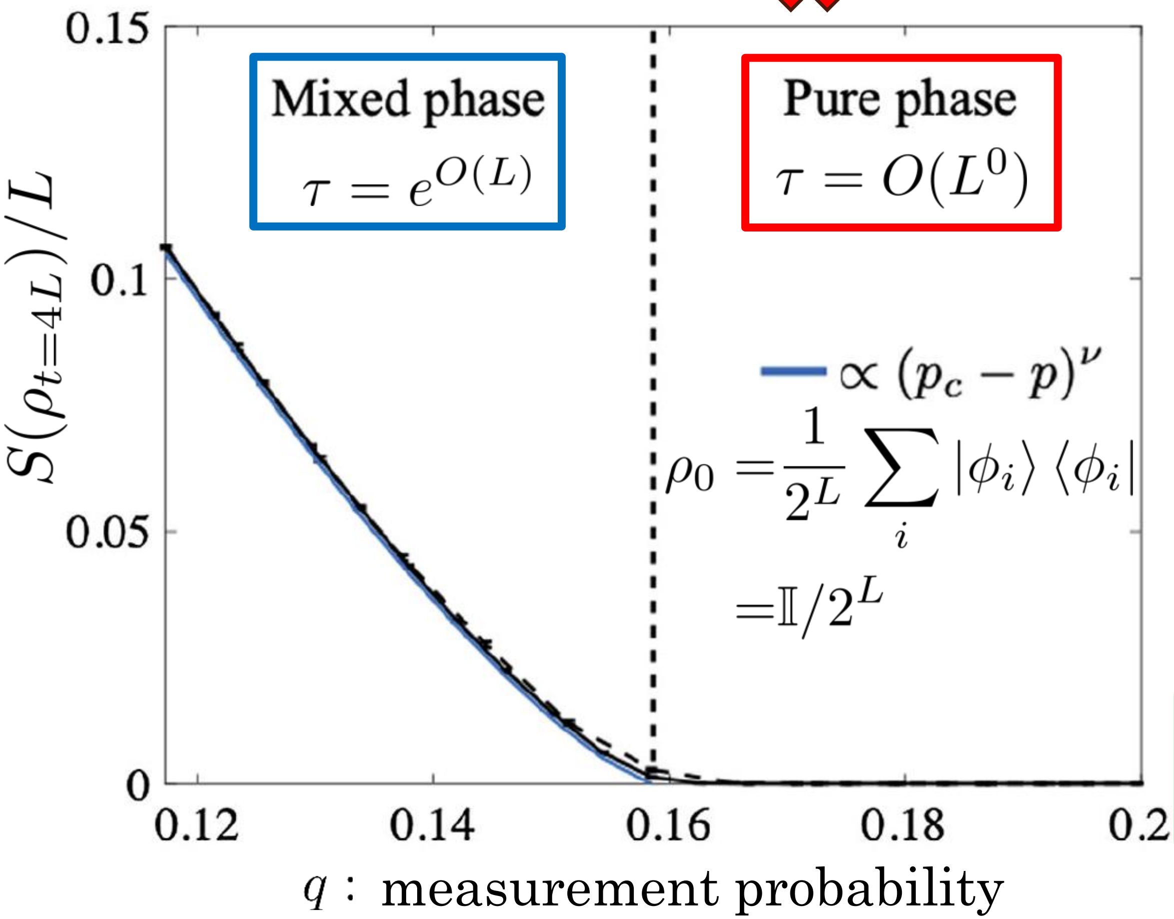


Transition of purification timescale

mixed-state dynamics :

circuit $\rho_t \to \mathsf{U}\rho_t\mathsf{U}^\dagger$

measurement $\rho_t \to \dfrac{\mathsf{M}_\omega \rho_t \mathsf{M}_\omega^\dagger}{P_{\rho_t}(\omega)}$



Born probability:

$$P_{\rho_t}(\omega) = \mathrm{tr}\left(\mathsf{M}_\omega \rho_t \mathsf{M}_\omega^\dagger\right)$$

purification at τ :

$$\rho_\tau \simeq |\psi_\tau\rangle\langle\psi_\tau|\,, \quad S(\rho_\tau) \simeq 0$$

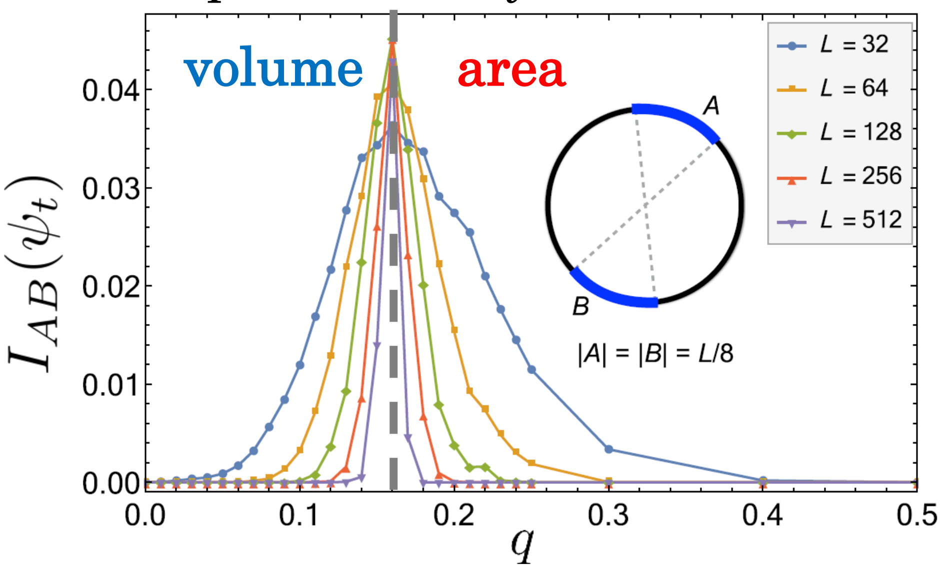
$$S(\rho) = -\mathrm{tr}\left[\rho \log(\rho)\right]$$

purification transition in the mixed-state dynamics

M. J. Gullans et.al., PRX **10**, 041020 (2020).

Mutual information and Purification timescale

pure-state dynamics



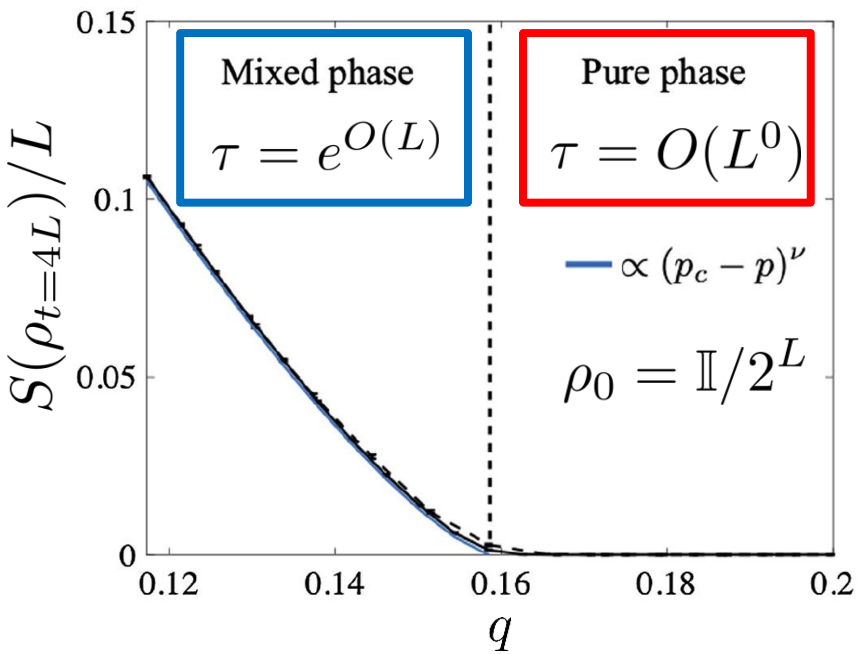
$$I_{AB}(\psi) = S_A(\psi) + S_B(\psi) - S_{AB}(\psi)$$

peak of $I_{AB}(\psi_t)$

entanglement transition

mixed-state dynamics

purification at $\tau : S(\rho_\tau) \simeq 0$



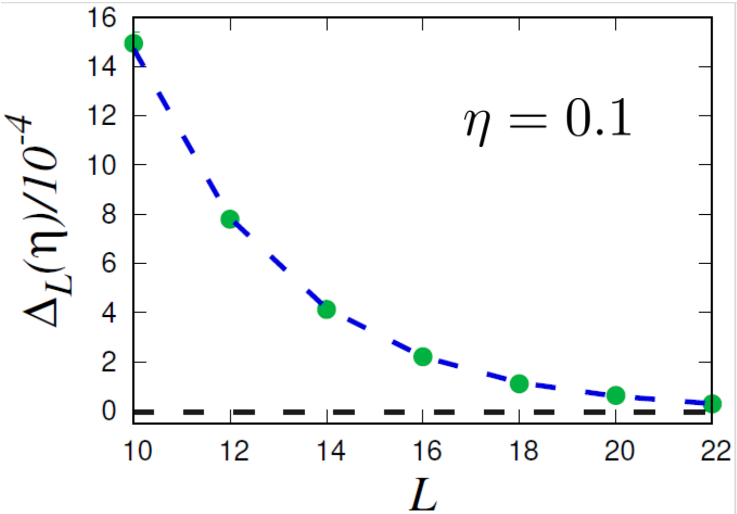
entanglement transition (pure)

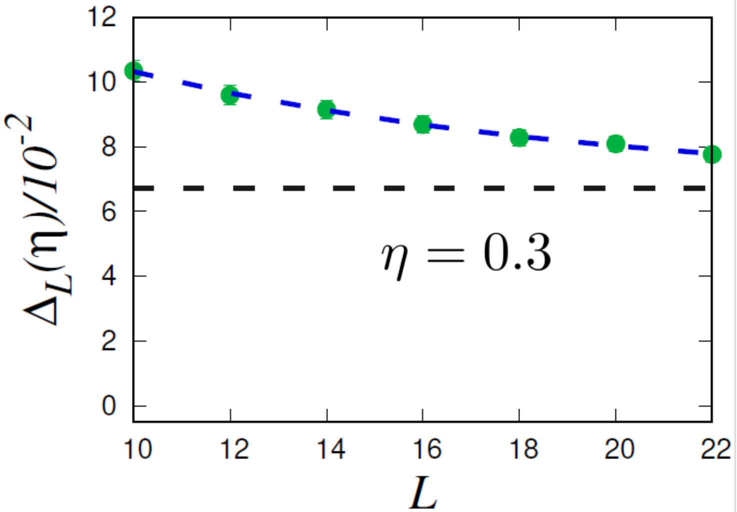
purification transition (mixed)

Y. Li et.al., PRB **100**, 134306 (2019).
M. J. Gullans et.al., PRX **10**, 041020 (2020).

Fitting

Ken Mochizuki
Ryusuke Hamazaki
PRL **134**, 010410 (2025).



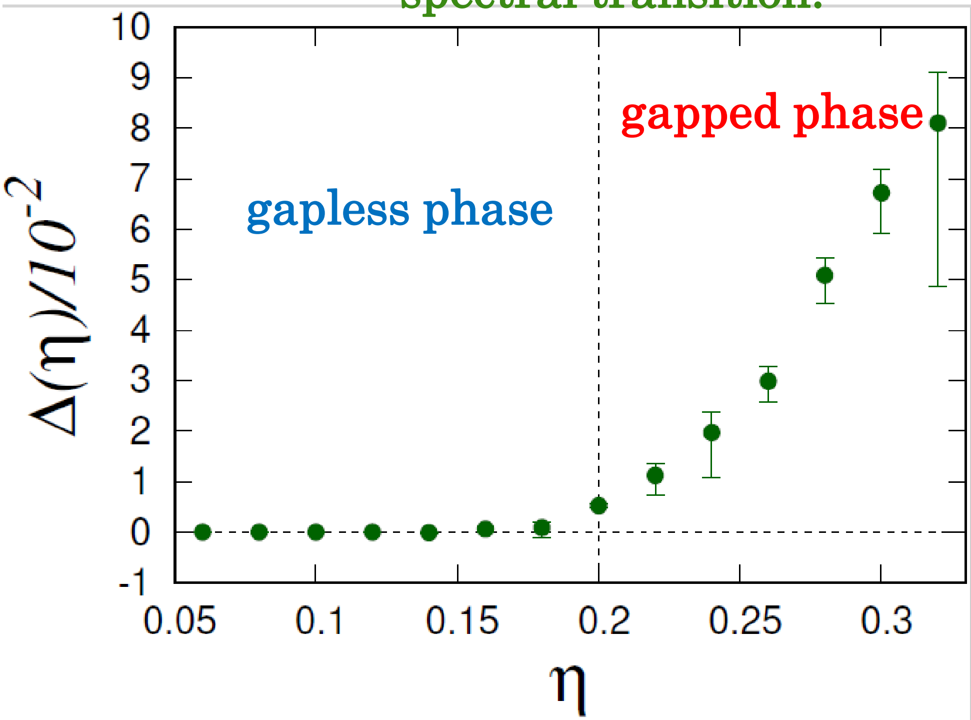


numerical data

$$\Delta_L^{\mathrm{fit}}(\eta) = \Delta(\eta) + \alpha(\eta)[\beta(\eta)]^{-L}$$

$$\Delta(\eta) = \lim_{L\to\infty} \Delta_L^{\mathrm{fit}}(\eta)$$

spectral transition!



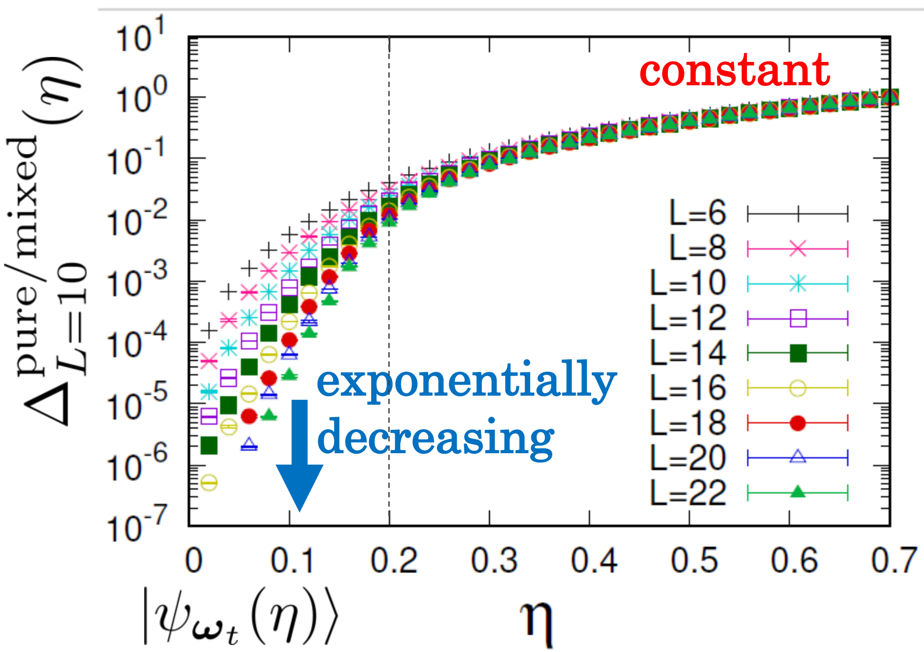
Spectral transition (pure-state) $\Leftrightarrow$ Purification transition (mixed-state)

mixed-state dynamics: $\rho_{\boldsymbol{\omega}_{t+1}}(\eta) = \dfrac{Q_{\omega_{t+1}}(\eta)\rho_{\boldsymbol{\omega}_t}(\eta)Q^\dagger_{\omega_{t+1}}(\eta)}{P_{\rho_{\boldsymbol{\omega}_t}(\eta)}(\omega_{t+1})}$

Born probability: $P_{\rho_{\boldsymbol{\omega}_t}(\eta)}(\omega_{t+1}) = \mathrm{tr}\left[Q_{\omega_{t+1}}(\eta)\rho_{\boldsymbol{\omega}_t}(\eta)Q^\dagger_{\omega_{t+1}}(\eta)\right]$

$$t \gg \tau_{\delta,L}(\eta) = 1/\Delta_L^{\mathrm{mixed}}(\eta) \to \rho_{\boldsymbol{\omega}_t}(\eta) \propto \underline{V_{\boldsymbol{\omega}_t}(\eta)\rho_0 V^\dagger_{\boldsymbol{\omega}_t}(\eta)} \simeq |\Psi^1_{\boldsymbol{\omega}_t}(\eta)\rangle\langle\Psi^1_{\boldsymbol{\omega}_t}(\eta)|$$

$\simeq$ rank-1 matrix $\Rightarrow$ purification



if $\boxed{\Delta_L^{\mathrm{mixed}}(\eta) = \Delta_L^{\mathrm{pure}}(\eta)}$

gapless phase $\Rightarrow$ mixed phase :

$$\Delta_L(\eta) = e^{-O(L)} \to \tau_{\delta,L}(\eta) = e^{O(L)}$$

gapped phase $\Rightarrow$ pure phase :

$$\Delta_L(\eta) = O(L^0) \to \tau_{\delta,L}(\eta) = O(L^0)$$

$$\rho_0 = \frac{1}{2^L}\sum_i |\phi_i\rangle\langle\phi_i| = \frac{\mathbb{I}}{2^L}$$

K. Mochizuki and R. Hamazaki, PRL **134**, 010410 (2025).

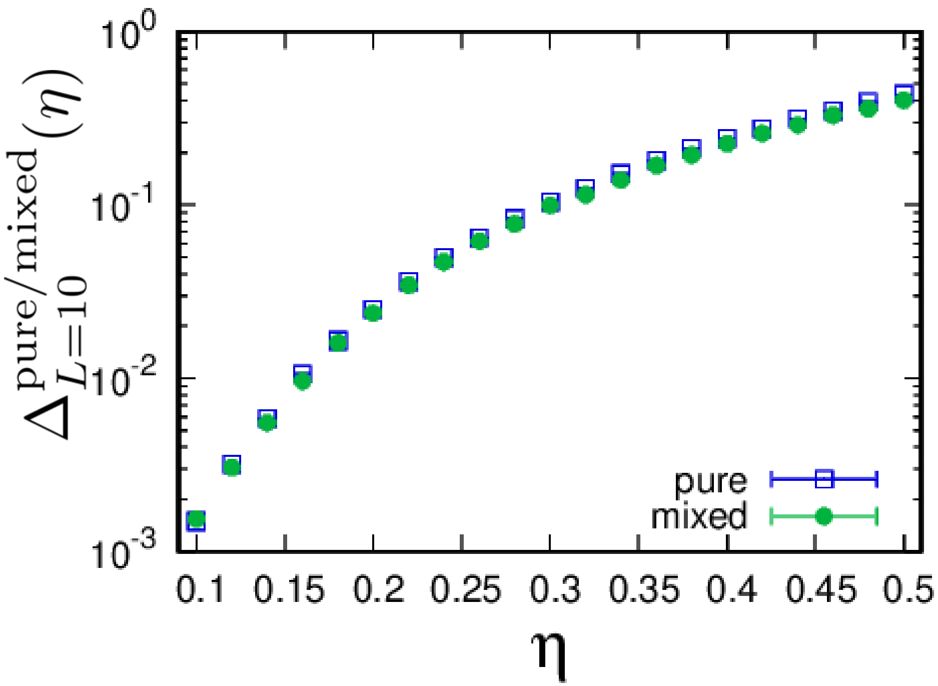
Spectral transition (pure-state) ⇔ Purification transition (mixed-state)

mixed-state dynamics: $\rho_{\boldsymbol{\omega}_{t+1}}(\eta) = \dfrac{Q_{\omega_{t+1}}(\eta)\rho_{\boldsymbol{\omega}_t}(\eta)Q^\dagger_{\omega_{t+1}}(\eta)}{P_{\rho_{\boldsymbol{\omega}_t}(\eta)}(\omega_{t+1})}$

Born probability: $P_{\rho_{\boldsymbol{\omega}_t}(\eta)}(\omega_{t+1}) = \mathrm{tr}\left[Q_{\omega_{t+1}}(\eta)\rho_{\boldsymbol{\omega}_t}(\eta)Q^\dagger_{\omega_{t+1}}(\eta)\right]$

$$t \gg \tau_{\delta,L}(\eta) = 1/\Delta_L^{\mathrm{mixed}}(\eta) \to \rho_{\boldsymbol{\omega}_t}(\eta) \propto \underline{V_{\boldsymbol{\omega}_t}(\eta)\rho_0 V^\dagger_{\boldsymbol{\omega}_t}(\eta)} \simeq |\Psi^1_{\boldsymbol{\omega}_t}(\eta)\rangle\langle\Psi^1_{\boldsymbol{\omega}_t}(\eta)|$$

≃ rank-1 matrix ⇒ purification



if $\boxed{\Delta_L^{\mathrm{mixed}}(\eta) = \Delta_L^{\mathrm{pure}}(\eta)}$

gapless phase ⇒ mixed phase：

$$\Delta_L(\eta) = e^{-O(L)} \to \tau_{\delta,L}(\eta) = e^{O(L)}$$

gapped phase ⇒ pure phase：

$$\Delta_L(\eta) = O(L^0) \to \tau_{\delta,L}(\eta) = O(L^0)$$

spectral transition

⇓

purification transition

K. Mochizuki and R. Hamazaki, PRL **134**, 010410 (2025).

Comparison of transitions

	Monitored dynamics where Hamiltonians can be non-local		Well-known equilibrium systems described by local Hamiltonians
Gapped phase	$S_{L/2} = O(L^0)$ $\Delta_L = O(L^0)$	same	$S_{L/2} = O(L^0)$ $\Delta_L = O(L^0)$
Gapless phase	$S_{L/2} = O(L)$ $\Delta_L = \exp\left[-O(L)\right]$	distinct	$S_{L/2} = O[\log(L)]$ $\Delta_L = O[1/\mathrm{poly}(L)]$

?

long-range interaction ⇒ volume law?

*1D systems
e.g. XXZ model

T. Udagawa and H. Katsura,
J. Phys. A: Math. Theor. **50** (2017) 405002.

B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen,
Quantum information meets quantum matter (Springer, 2019).

Ken Mochizuki, Ryusuke Hamazaki, Physical Review Letters **134**, 010410 (2025).

Irreducibility

Hironobu Yoshida
PRA **109**, 022218 (2024).

averaged dynamics: $\rho_{\tau+1} = \Gamma(\rho_\tau) = \sum_\omega Q(\omega)\rho_\tau Q^\dagger(\omega)$

- irreducibility: any operator O can be constructed as $O = \sum_{t,\boldsymbol{\omega}_t} c_{t,\boldsymbol{\omega}_t} V(\boldsymbol{\omega}_t)$

➡ **positive-definite** (a) and **unique** (b) stationary state in averaged dynamics

positive-**semi**-definite stationary state (always present): $\rho_\infty = \Gamma(\rho_\infty) \geq 0$

a. zero eigenvalue $\rho_\infty |\phi_0\rangle = 0 \Rightarrow \rho_\infty = 0$

b. positive-definite stationary states $\rho_\infty^1, \rho_\infty^2 \Rightarrow \left(\sum_i c_i \rho_\infty^i\right)|\phi_0(\boldsymbol{c})\rangle = 0$

proof-a: $\langle\phi_0|\rho_\infty|\phi_0\rangle = \langle\phi_0|\Gamma^t(\rho_\infty)|\phi_0\rangle$

$$= \sum_{\boldsymbol{\omega}_t} \langle\phi_0| V(\boldsymbol{\omega}_t)\rho_\infty V^\dagger(\boldsymbol{\omega}_t)|\phi_0\rangle = \sum_{\boldsymbol{\omega}_t} \left|\sqrt{\rho_\infty}V^\dagger(\boldsymbol{\omega}_t)|\phi_0\rangle\right|^2 = 0$$

$\rightarrow \rho_\infty V^\dagger(\boldsymbol{\omega}_t)|\phi_0\rangle = 0$ for any $t, \boldsymbol{\omega}_t$ $\rightarrow \rho_\infty O|\phi_0\rangle = 0$ for any O

proof-b: $\rho_\infty(y) = (1-y)\rho_\infty^1 - y\rho_\infty^2$

$\rightarrow \rho_\infty(y^*)|\phi_0(y^*)\rangle = 0$

$\rho_\infty(y)|\phi_i(y)\rangle = \lambda_i(y)|\phi_i(y)\rangle$