空間力学系から理解する非線形トポロジカル絶縁体

曽根和樹

筑波大学　数理物質系

Refs. K. Sone, M. Ezawa, Y. Ashida, N. Yoshioka, and T. Sagawa, Nat. Phys. 20, 1164 (2024).
K. Sone, M. Ezawa, Z. Gong, T. Sawada, N. Yoshioka, and T. Sagawa, Nat. Commun. 16, 422 (2025).
K. Sone and Y. Hatsugai, arXiv:2501.10087.

2025/3/25　統計物理学懇談会

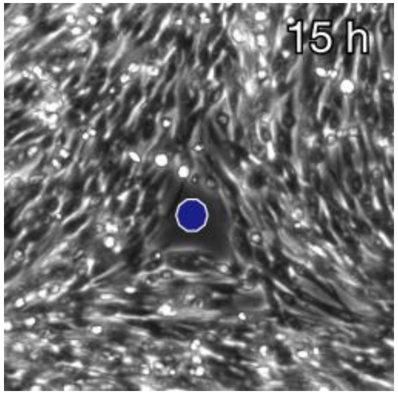
Outline

- Introduction & Overview
- Definition of nonlinear bulk-edge correspondence
- Numerical demonstration
- Analysis from nonlinear transfer matrices
- Bifurcation and nonlinear topological phenomena
- Further extension

Topology in Physics

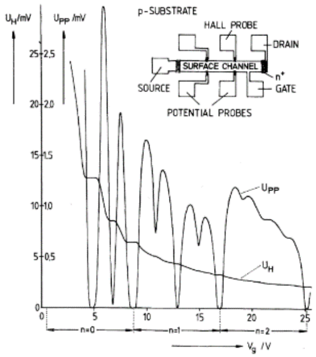
Real space

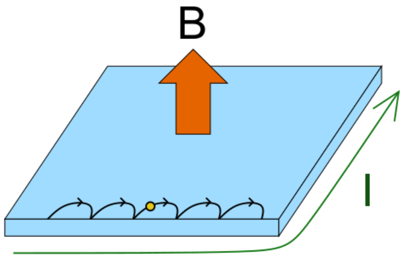
S. Heinze et al., Nat. Phys. 7, 713 (2011).
A. Fert, N. Reyren, V. Cros, Nat. Rev. Mat. 2, 17031 (2017).



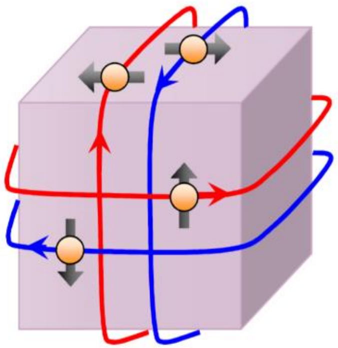
K. Kawaguchi, R. Kageyama, M. Sano, Nature 545, 328 (2017).

Wavenumber space (band structure)





K. v. Klitzing et al., Phys, Rev. Lett. 45, 494 (1980).
D. J. Thouless *et al.*, Phys. Rev. Lett. 49, 405 (1982).



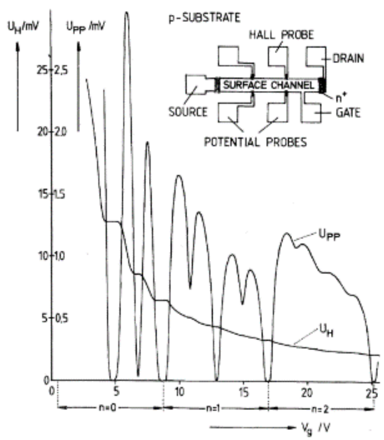
L. Fu, C. L. Kane, and E. J. Mele,
Phys. Rev. Lett. 98, 106803 (2007).
Y. Ando, J. Phys. Soc. of Jpn. 82, 102001 (2013).

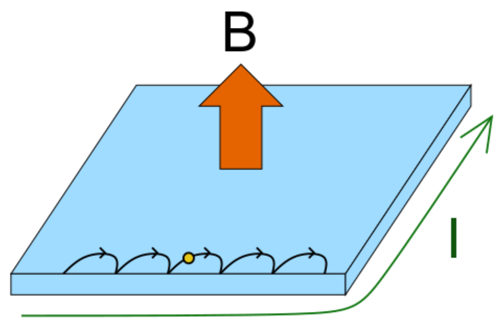
My studies

Typical Example: Quantum Hall Effect

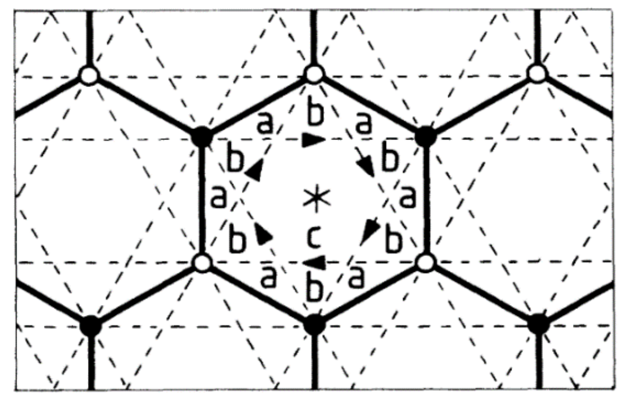
Quantum Hall effect
(under external magnetic field)

Anomalous quantum Hall effect
(utilizing local magnetic flux)





K. v. Klitzing et al., Phys, Rev. Lett. 45, 494 (1980)
D. J. Thouless *et al.*, Phys. Rev. Lett. 49, 405 (1982)



F. D. M. Haldane Phys, Rev. Lett. 61, 2015 (1988)
K. Ohgushi *et al.* Phys. Rev. B 62, R6065 (2000)

- Quntized Hall current ⇔ Edge current
  ⇔ Bulk topological invariant

$$\sigma_{xy} = \frac{e^2}{h}\nu$$

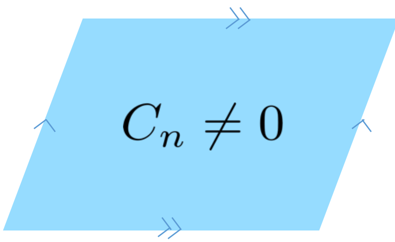
ν: sum of Chern numbers of the occupied bands

Band Topology and Bulk-Edge Correspondence

Bulk (periodic boundary)

$$H(\mathbf{k})|\psi_n(\mathbf{k})\rangle = E_n(\mathbf{k})|\psi_n(\mathbf{k})\rangle$$

(Eigenequation of each wavenumber sector)



Chern number

$$C_n = \frac{1}{2\pi}\int_{\mathrm{BZ}} \nabla_{\mathbf{k}} \times \mathbf{A}_n(\mathbf{k}) \cdot d\mathbf{S}$$

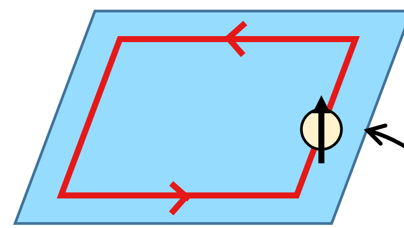
$$\mathbf{A}_n(\mathbf{k}) = i\langle\psi_n(\mathbf{k})|\nabla_{\mathbf{k}}|\psi_n(\mathbf{k})\rangle$$

Bulk-edge correspondence

$\Leftrightarrow$

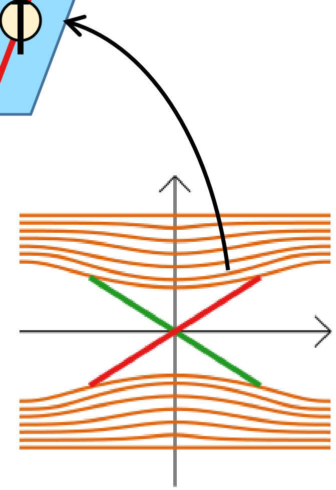
Edge (open boundary)

$$H|\Psi_n\rangle = E_n|\Psi_n\rangle$$



Edge modes

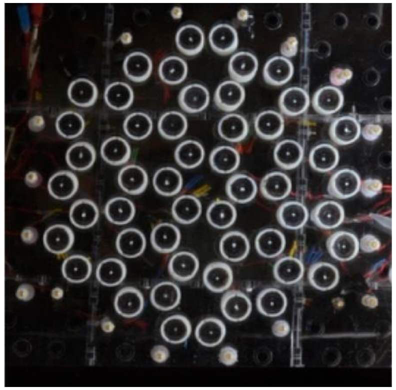
- **Localized** states
- Gapless dispersions
- **Backscattering-free** (unidirectional) current
- **Robustness** against disorders



Topology in Classical Systems

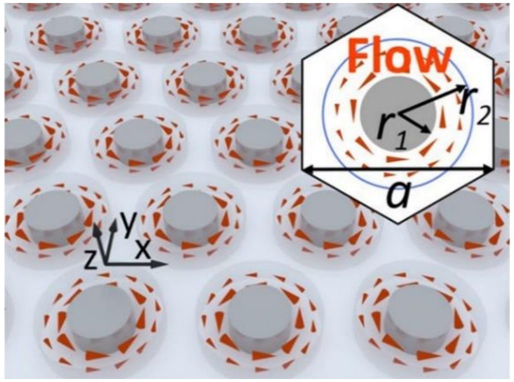
Condensed matter	Classical system
Wave functions for eigenstates	Normal modes of physical quantities (e.g. location)
Hamiltonian	Coefficient matrix in linear dynamics
$H\Psi = E\Psi$	$\mathcal{H}\vec{X} = \omega\vec{X}$

Mechanical lattice



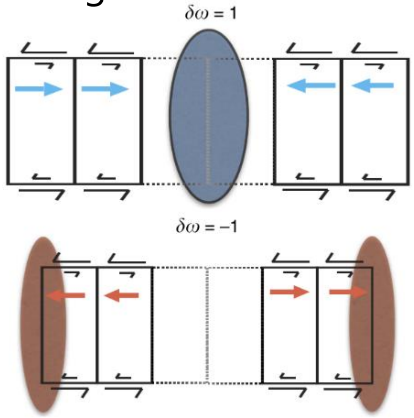
L. M. Nash et al.,
PNAS 112, 14495 (2015).

Fluid



Z. Yang et al. PRL 114, 114301 (2015).

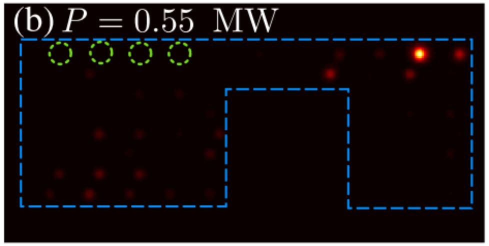
Stochastic process
(Biological kinetic network)



A, Murugan and S. Vaikuntanathan
Nat. Commun. 8, 13881 (2017).

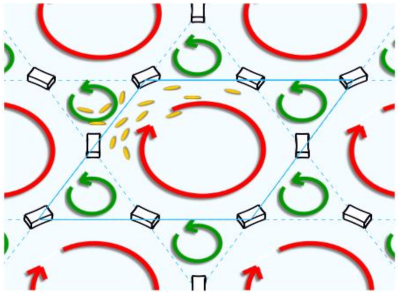
Platforms to Study Nonlinear Topology

Photonics



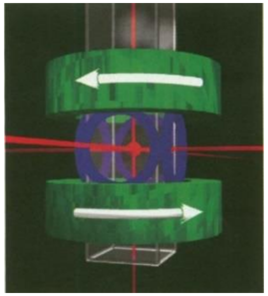
D. Leykam and Y. D. Chong PRL 117, 143901 (2016).

Active matter



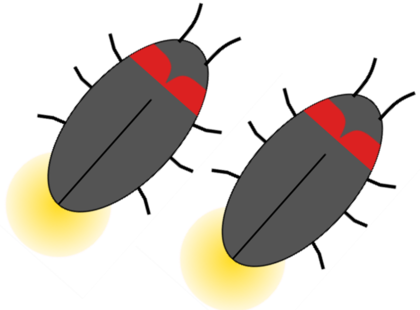
K. Sone and Y. Ashida PRL 123, 205502 (2019).

Ultracold atoms



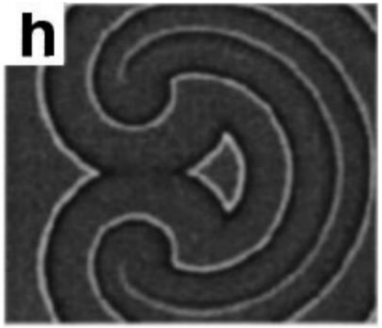
M. H. Anderson *et al.* Science 269, 198 (1995).

Biological oscillators



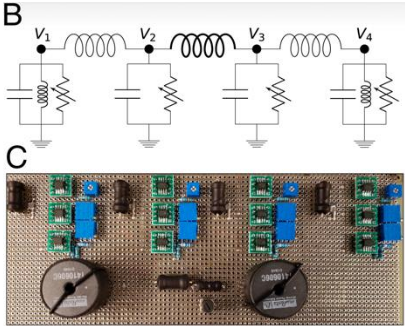
J. Buck, Quart. Rev. Biol. 63, 265–289 (1988).

Chemical reactions



T. Amemiya *et al.* Chaos. 8, 872 (1998).

Electrical circuits



T. Kotwal *et al.* PNAS 118, e2106411118 (2019).

Possible Nonlinear Topological Phenomena

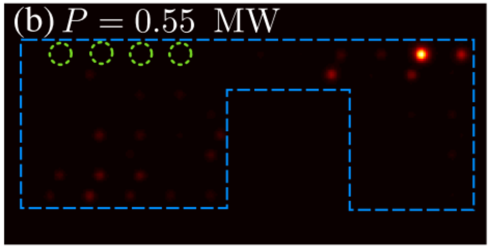
Unique phenomena

- Self-bulk-localized modes
  Y. Lumer et al. PRL 111, 243905 (2013)

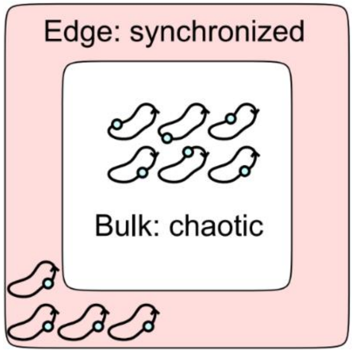
- Edge soliton
  D. Leykam and Y. D. Chong PRL 117, 143901 (2016)
  Z. Zhang et al. Nat. Commun. 11, 1902 (2020)

- Topological synchronization
  K. Sone, Y. Ashida, and T. Sagawa
  PRReserch 4, 023211 (2022)
  F. Di, W. Zhang, and X. Zhang
  Commun. Phys. 8, 78 (2025)

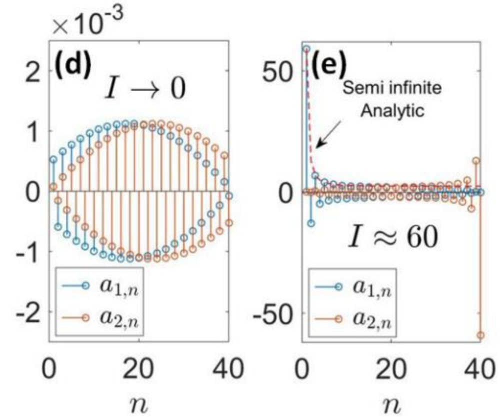
- Amplitude dependence
  = Nonlinearity-induced transition
  Y. Hadad et al. PRB 93, 155112 (2016)
  D. Zhou et al. Nat. Commun. 13, 3379 (2022)



D. Leykam and Y. D. Chong PRL 117, 143901 (2016)



K. Sone, Y. Ashida, and T. Sagawa PRR 4, 023211 (2022)



Y. Hadad et al. PRB 93, 155112 (2016)

Main Problems

Q1. Bulk-edge correspondence
(topological invariant, edge mode)

Linear

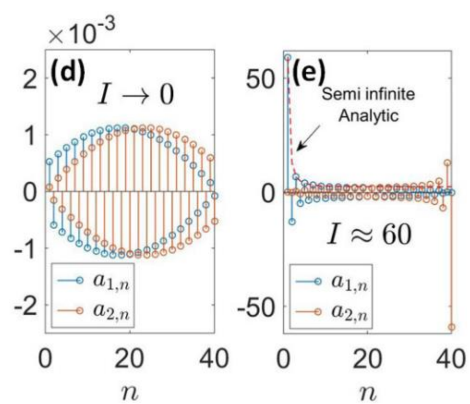
$$C_n = \frac{1}{2\pi}\int_{\mathrm{BZ}} \nabla_{\mathbf{k}} \times \mathbf{A}_n(\mathbf{k}) \cdot d\mathbf{S}$$

$$\mathbf{A}_n(\mathbf{k}) = i\langle \psi_n(\mathbf{k})|\nabla_{\mathbf{k}}|\psi_n(\mathbf{k})\rangle$$

→ **?**

Q2. (Simple) analytical techniques

Q3. Origin of nonlinear phenomena
(cf. nonlinearity-induced transition)



Overview

Q1. Bulk-edge correspondence (topological invariant, edge mode)

Linear

$$C_n = \frac{1}{2\pi}\int_{\mathrm{BZ}} \nabla_{\mathbf{k}} \times \mathbf{A}_n(\mathbf{k}) \cdot d\mathbf{S}$$

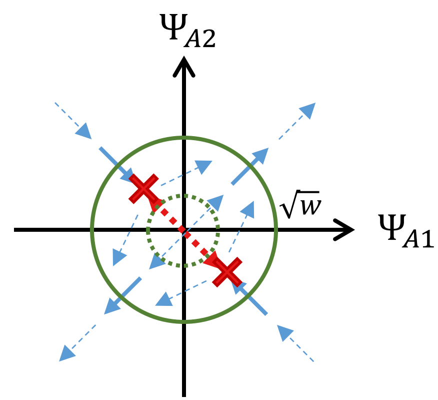
$$\mathbf{A}_n(\mathbf{k}) = i\langle \psi_n(\mathbf{k})|\nabla_{\mathbf{k}}|\psi_n(\mathbf{k})\rangle$$

→

A1. Defined by nonlinear eigenvalue problems

$$f_j(\boldsymbol{k}, w; \psi) = E(\boldsymbol{k}, w)\psi_j(\boldsymbol{k}, w)$$

$$C_{\mathrm{NL}}(w) = \frac{1}{2\pi i w}\int \nabla \times \langle \psi(\boldsymbol{k}, w)|\nabla|\psi(\boldsymbol{k}, w)\rangle d^2\boldsymbol{k}$$

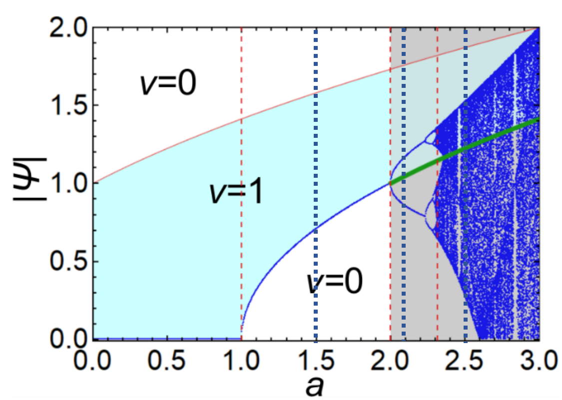


Q2. (Simple) analytical techniques

A2. Nonlinear transfer matrix (spatial dynamics)

Q3. Origin of nonlinear phenomena (cf. nonlinearity-induced transition)

A3. Bifurcations and chaos

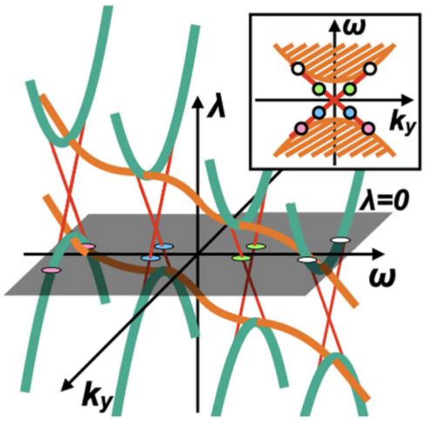


What I Do NOT Talk About

- **Dynamical (temporal) stability or instability**
  → Irrelevant to the bulk-edge correspondence
  (cf. Non-Hermitian edge modes can have dynamical instability.)

- **Nonlinearity of eigenvalues**
  (Different physical setup)

	Isobe-Yoshida-Hatsugai	Our work
Nonlinearity	Eigenvalue	Eigenvector
Origin of nonlinearity	Higher-order differential equations or Frequency-dependent response functions	Nonlinear dynamics or Mean-field of many-body interactions



T. Isobe, T. Yoshida, Y. Hatsugai PRL 132, 126601 (2024).

Outline

- Introduction & Overview
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Setup: Conservative Nonlinear Dynamics with U(1) Symmetry

General nonlinear dynamics

$$i\frac{\partial \Psi_j(\boldsymbol{r})}{\partial t} = f_j(\Psi;\boldsymbol{r})$$

cf. Possible extension

D. Zhou et al., Nat. Commun. 13, 3379 (2022)

D. Zhou New J. Phys. 26, 073009 (2024)

Assumptions

- $U(1)$ symmetry, $f_j\left(e^{i\theta}\Psi;\boldsymbol{r}\right) = e^{i\theta} f_j(\Psi;\boldsymbol{r})$
- Conservative dynamics, $\sum_{i,\boldsymbol{r}} |\Psi(\boldsymbol{r})|^2$ = constant (≃Hermitian)

c.f. Gross-Pitaevskii equation (ultracold atoms)

$$i\frac{\partial \Psi(\boldsymbol{r})}{\partial t} = f(\Psi;\boldsymbol{r}) = -\frac{\nabla^2\Psi(\boldsymbol{r})}{2m} + V\Psi(\boldsymbol{r}) + \frac{4\pi a}{\mathrm{m}}|\Psi(\boldsymbol{r})|^2\Psi(\boldsymbol{r})$$

c.f. Kerr-type nonlinearity in lattice systems (such as photonics)

$$i\frac{\partial \Psi_j(\boldsymbol{r})}{\partial t} = f_j(\Psi;\boldsymbol{r}) = \sum_{k,\boldsymbol{r}'} H_{jk}(\boldsymbol{r},\boldsymbol{r}')\Psi_k(\boldsymbol{r}') + \kappa_j\left|\Psi_j(\boldsymbol{r})\right|^2\Psi_j(\boldsymbol{r})$$

Nonlinear Eigenvalue Problem

$$i\frac{\partial \Psi_j(\boldsymbol{r})}{\partial t} = f_j(\Psi;\boldsymbol{r})$$

Assuming temporal periodicity $\Psi_j(\boldsymbol{r};t) = e^{-iEt}\Psi_j(\boldsymbol{r})$

Nonlinear eigenvalue problem

cf. T. Tulop et al., PRB 102, 115411 (2020)
D. Zhou et al., Nat. Commun. 13, 3379 (2022)

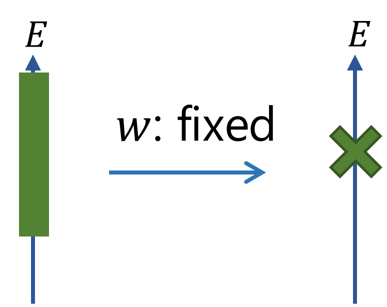
$$f_j(\Psi;\boldsymbol{r}) = E\Psi_j(\boldsymbol{r})$$

(Ψ: Nonlinear eigenvector)
(E: Nonlinear eigenvalue)

e.g. $$i\frac{d\psi}{dt} = \psi + |\psi|^2\psi \qquad f(\psi) = (1+|\psi|^2)\psi$$

Eigenvector: $\psi = \sqrt{w}$ Eigenvalue: $E = 1 + w$

Dependence on the amplitude w
(absence of the superposition principle)
$\Rightarrow$ **w is an additional parameter.**



Discrete spectrum (band)

Nonlinear Chern Number

Real-space description of the nonlinear eigenequation

$f_j(\Psi; \boldsymbol{r}) = E\Psi_j(\boldsymbol{r})$

↓

Assuming the **Bloch ansatz** $\Psi(w; \boldsymbol{r} + \boldsymbol{a}_j) = e^{ik_j}\Psi(w; \boldsymbol{r})$ ($\boldsymbol{a}_j$: lattice vector)

Focus on special solutions with $w = \sum_i |\psi_i(\boldsymbol{k})|^2$ **being independent of $\boldsymbol{k}$**

Wavenumber-space description of the nonlinear eigenequation

$f_j\big(\boldsymbol{k}, w; \psi(\boldsymbol{k}, w)\big) = E(\boldsymbol{k}, w)\psi_j(\boldsymbol{k}, w)$

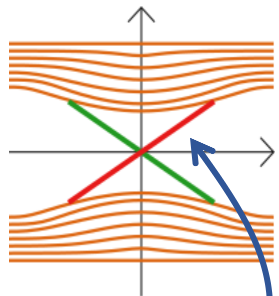
Nonlinear topological invariant (nonlinear Chern number)

$$C_{\mathrm{NL}}(w) = \frac{1}{2\pi i w} \int \nabla \times \langle \psi(\boldsymbol{k}, w) | \nabla | \psi(\boldsymbol{k}, w) \rangle d^2\boldsymbol{k}$$

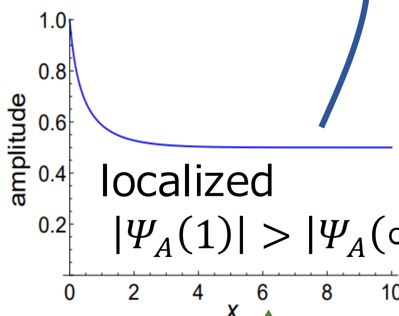
- Quantized as in linear cases
- w is an additional parameter.
- w dependence ⇒ **nonlinearity-induced topological phase transition**

Correspondence to Localization vs. Anti-Localization

$C_{\mathrm{NL}}(w) \neq 0$



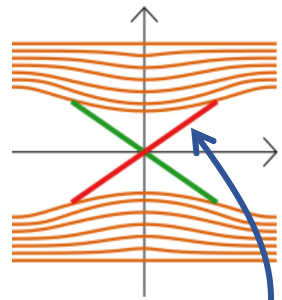
Fix $|\Psi_A(0)| = w$

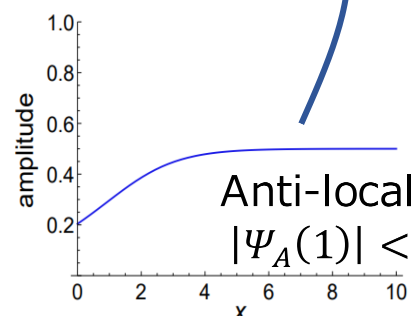


localized

$|\Psi_A(1)| > |\Psi_A(\infty)|$

$C_{\mathrm{NL}}(w) = 0$

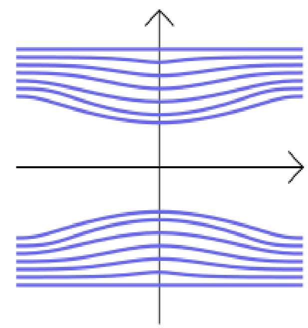




Anti-localized

$|\Psi_A(1)| < |\Psi_A(\infty)|$

or

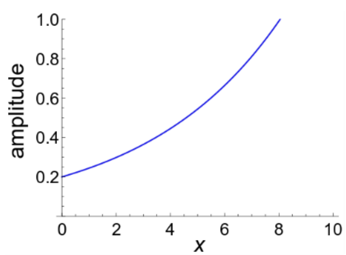


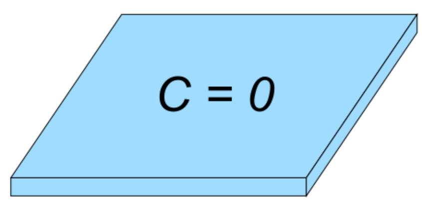
No zero modes

Adiabatically removed

(connected to linear trivial phase)

Divergence at $x \to \infty$
(disappearance)





Possible definition of
a nonlinear trivial phase

Mathematically shown in weakly nonlinear cases or local sense

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Nonlinear QWZ Model: Model of Nonlinear Chern Insulator

Kerr nonlinearity

Topological Hamiltonian (QI-Wu-Zhang model)

$$i\frac{d}{dt}\Psi_j(\boldsymbol{x}) = -\kappa(-1)^j\sum_k|\Psi_k(\boldsymbol{x})|^2\,\Psi_j(\boldsymbol{x}) + \sum_{k,\boldsymbol{x}} H_{jk}(\boldsymbol{x},\boldsymbol{x}')\Psi_k(\boldsymbol{x}') = \sum_{k,\boldsymbol{x}}\widetilde{H}_{jk}(\boldsymbol{x},\boldsymbol{x}';\Psi)\Psi_k(\boldsymbol{x}')$$

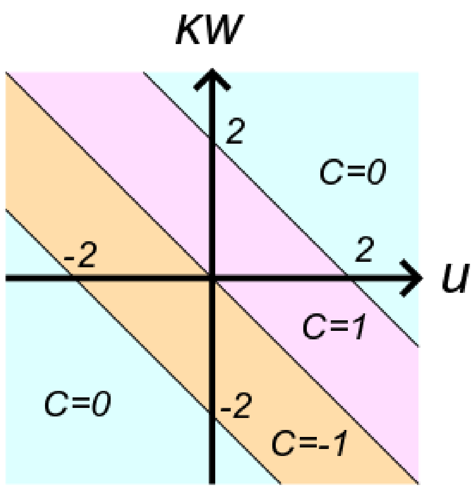
$$\widetilde{H}_{jk}(\boldsymbol{k};\Psi) = \begin{pmatrix} u+\kappa(\|\Psi\|_2)^2+\cos k_x+\cos k_y & \sin k_x - i\sin k_y \\ \sin k_x + i\sin k_y & -u-\kappa(\|\Psi\|_2)^2-\cos k_x-\cos k_y \end{pmatrix}$$

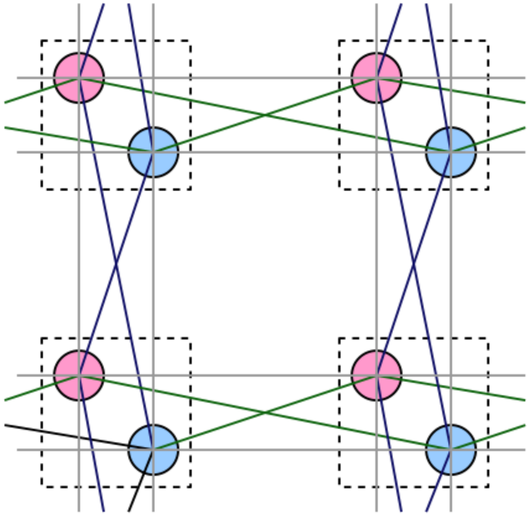
Ref. QI-Wu-Zhang model

X. L. Qi, Y. S. Wu, & S. C. Zhang PRB 74, 085308 (2006).

Phase diagram

(analytically obtained)



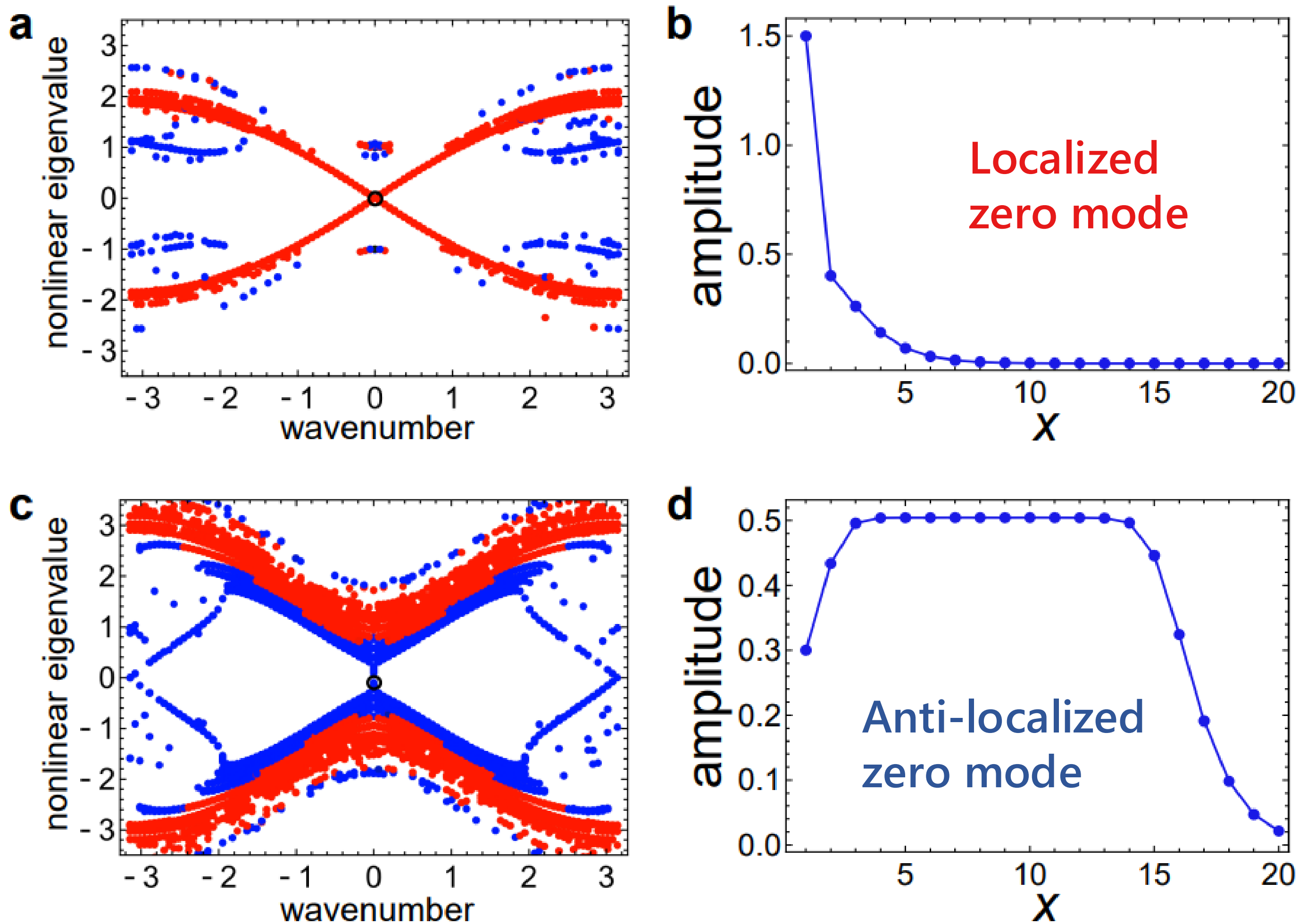


Bulk-Boundary Correspondence in Finite Lattice Systems

Band structures (x: open, y: periodic) **Eigenvectors**

$C_{\mathrm{NL}} = -1$

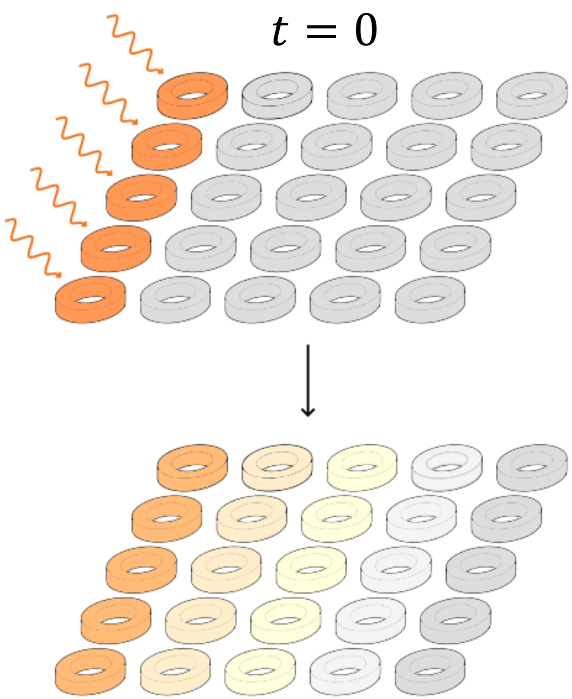
$C_{\mathrm{NL}} = 0$

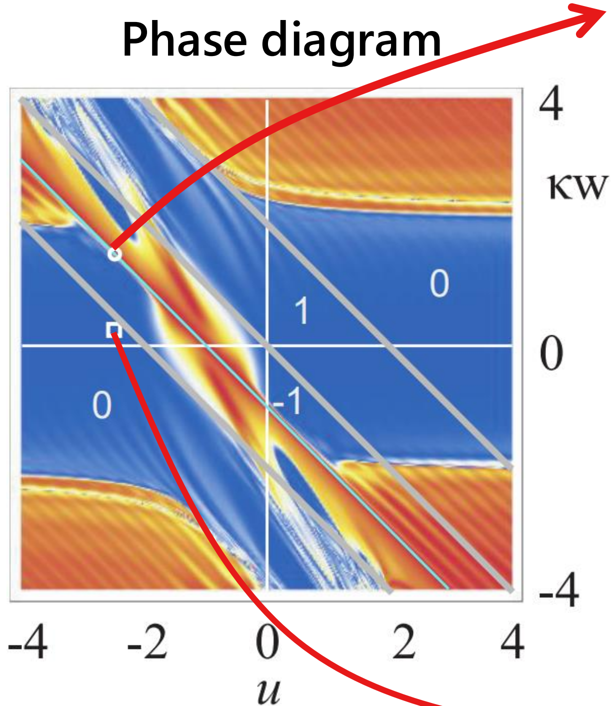


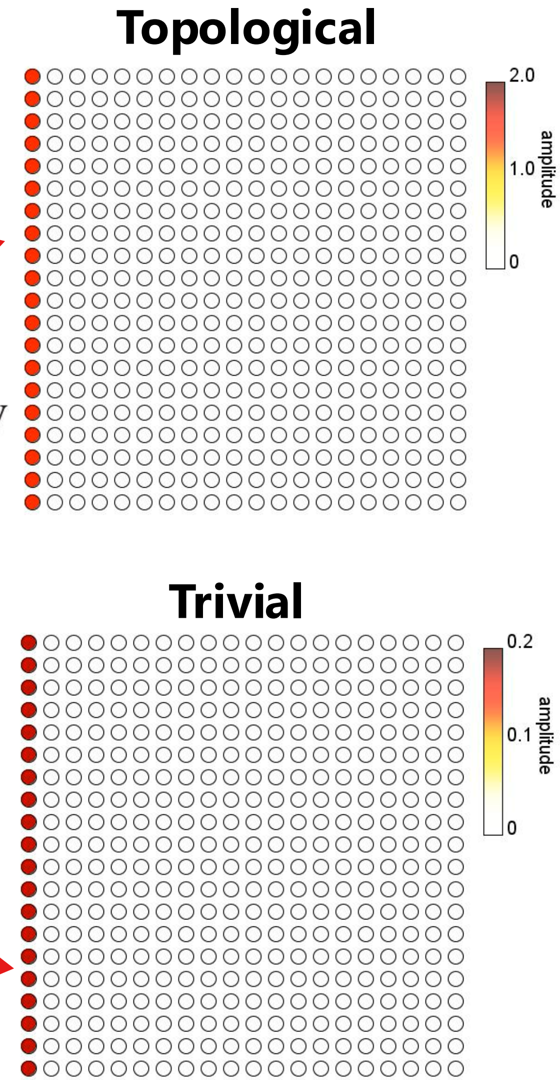
Red: localized, blue: delocalized

Observation by Quench Dynamics

Quench dynamics
= Time evolution from edge excitations







Calculated by Motohiko Ezawa sensei

Nonlinear Dirac Hamiltonian (Continuum Model)

Nonlinear Dirac Hamiltonian (effective low-energy model)

$$H_{\mathrm{Dirac}}(\Psi) = \begin{pmatrix} m + \kappa(\|\Psi\|_2)^2 & -i\partial_x + \partial_y \\ -i\partial_x - \partial_y & -m - \kappa(\|\Psi\|_2)^2 \end{pmatrix}$$

↓

Ansatz of edge modes $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = e^{ik_y y}\phi(x)\begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$, $E = k$

(Continuum) dynamical system describing gapless modes

$$\partial_x \phi(x) = m\phi(x) + \kappa|\phi(x)|^2\phi(x)$$

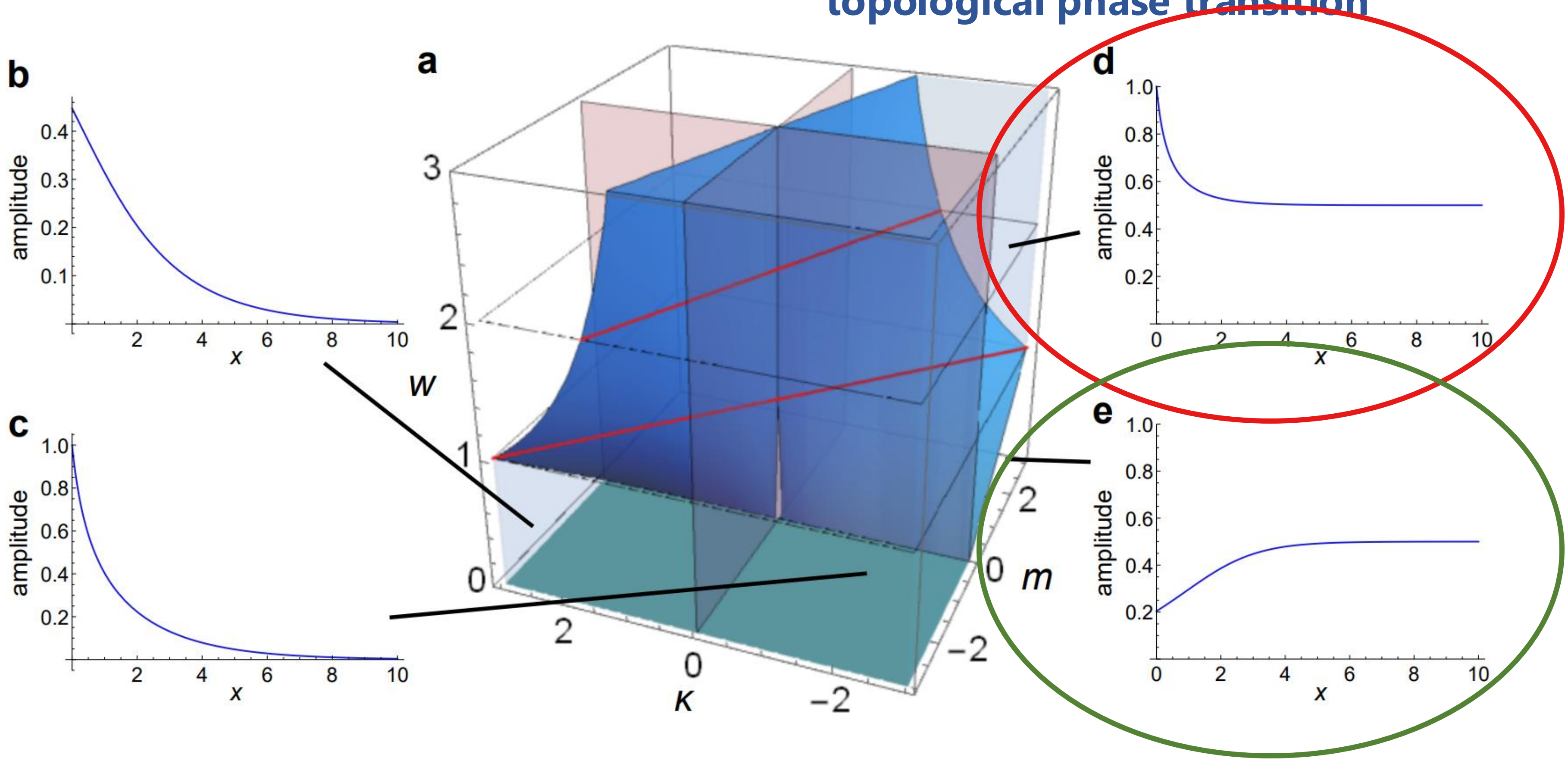
$$\Rightarrow \; \phi(x) = e^{i\theta}\sqrt{\frac{1}{\boxed{-\frac{\kappa}{m} + De^{-2mx}}}}$$

Positive

→ **Analytically confirm the bulk-edge correspondence**

Phase Diagram of Nonlinear Dirac Hamiltonian

Nonlinearity-induced topological phase transition



Anti-localized mode

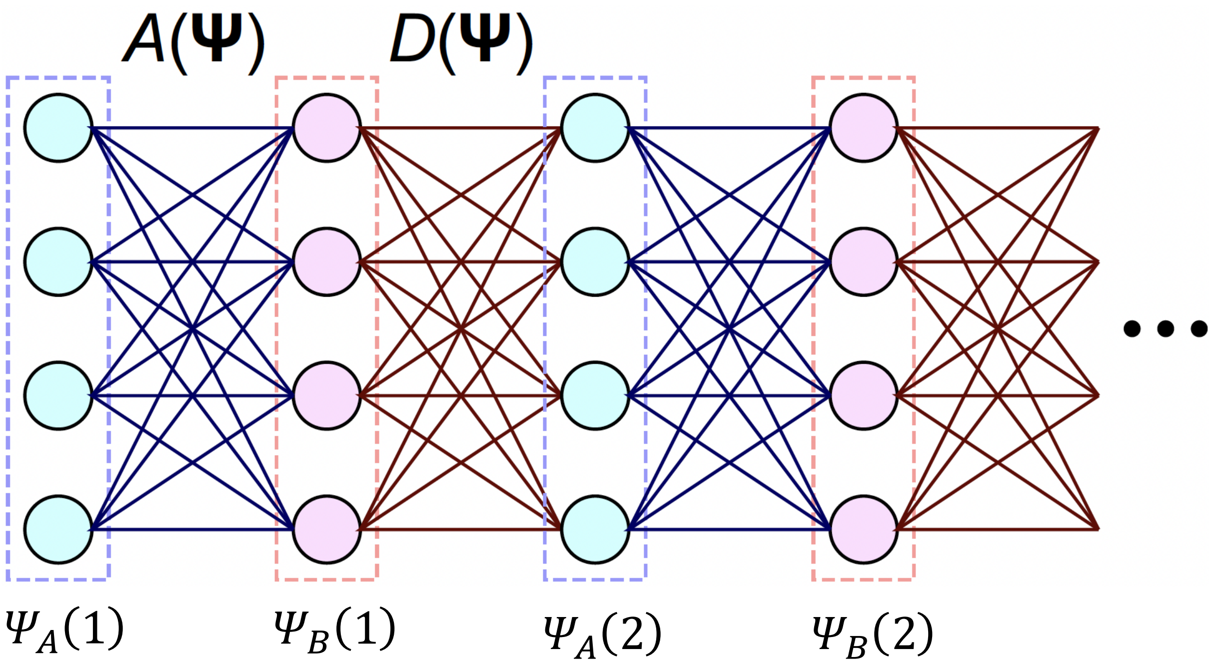
$C_{\mathrm{NL}} = 1/2 \Leftrightarrow$ Left-localized edge modes

Bulk-edge correspondence

Outline

- Introduction & Overview
- Definition of nonlinear bulk-edge correspondence
- Numerical demonstration
- Analysis from nonlinear transfer matrices
- Bifurcation and nonlinear topological phenomena
- Further extension

Nonlinear Transfer Matrix: 1D Sublattice Symmetric Case



Eigenvalue problem

$$E\Psi_A(x) = A^\dagger(\Psi)\Psi_B(x) + D^\dagger(\Psi)\Psi_B(x-1)$$

$$E\Psi_B(x) = A(\Psi)\Psi_A(x) + D(\Psi)\Psi_A(x+1)$$

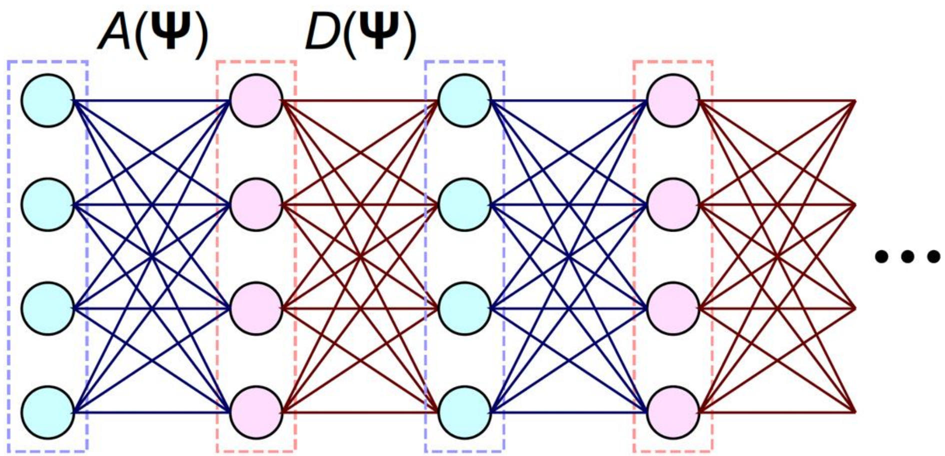
$\Updownarrow$

Transfer matrix (spatial dynamics)

$$\Psi_A(x+1) = ED(\Psi)^{-1}\Psi_B(x) - D(\Psi)^{-1}A(\Psi)\Psi_A(x)$$

$$\Psi_B(x+1) = EA^\dagger(\Psi)^{-1}\Psi_A(x+1) - A^\dagger(\Psi)^{-1}D^\dagger(\Psi)\Psi_B(x)$$

Nonlinear Transfer Matrix of Zero Modes



Sublattice symmetry → $E = 0$ for edge modes

Transfer matrix (spatial dynamics)

$$\Psi_A(x+1) = ED(\Psi)^{-1}\Psi_B(x) - D(\Psi)^{-1}A(\Psi)\Psi_A(x)$$

$$\Psi_B(x+1) = EA^{\dagger}(\Psi)^{-1}\Psi_A(x+1) - A^{\dagger}(\Psi)^{-1}D^{\dagger}(\Psi)\Psi_B(x)$$

Boundary condition $\Psi_B(0) = 0$

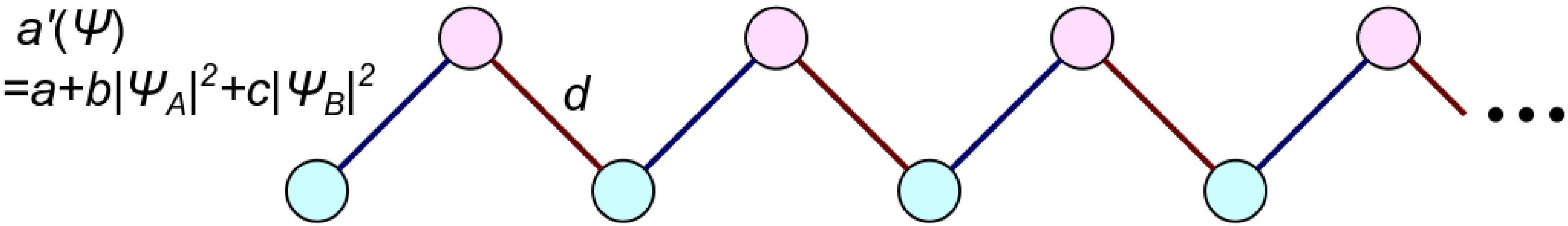
$E = 0$ ↓

Transfer matrix of zero mode

$$\Psi_A(x+1) = -D(\Psi)^{-1}A(\Psi)\Psi_A(x)$$

$$\Psi_B(x) = 0$$

Example: Nonlinear SSH Model (Nonlinear Hopping)



$$E\Psi_A(x) = (a + b|\psi_A(x)|^2 + c|\psi_B(x)|^2)\Psi_B(x) + d\Psi_B(x-1)$$
$$E\Psi_B(x) = (a + b|\psi_A(x)|^2 + c|\psi_B(x)|^2)\Psi_A(x) + d\Psi_A(x+1)$$

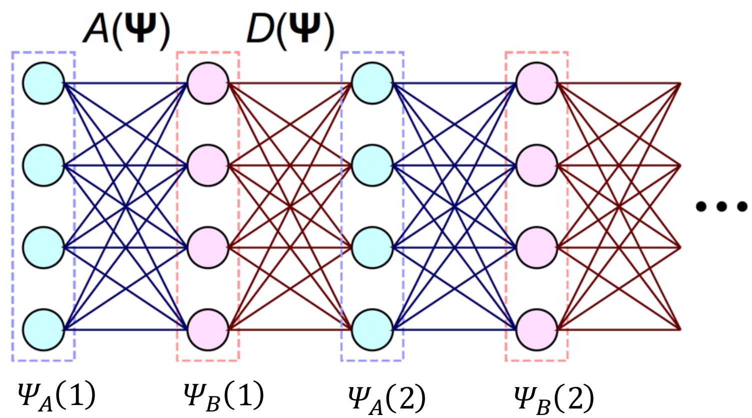
$E = 0$, semi-infinite system ($x > 0$)

$$\Psi_A(x+1) = -\frac{a + b|\Psi_A(x)|^2}{d}\Psi_A(x), \qquad \Psi_B(x) = 0$$

Note: Nonlinear SSH model ⟺ Nonlinear QWZ model at $k_y = 0$

$$H_{SSH}(k;\psi) = UH_{QWZ}(k,0;\psi)U^{-1}, U = (\sigma_y + \sigma_z)/\sqrt{2}$$

Nonlinear Winding Number



$$E\Psi_A(x) = A^\dagger(\Psi)\Psi_B(x) + D^\dagger(\Psi)\Psi_B(x-1)$$

$$E\Psi_B(x) = A(\Psi)\Psi_A(x) + D(\Psi)\Psi_A(x+1)$$

Wavenumber-space description

$$E\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = \begin{pmatrix} 0 & A^\dagger(\psi) + e^{-ik}D^\dagger(\psi) \\ A(\psi) + e^{ik}D(\psi) & 0 \end{pmatrix}\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$$

Assume $A(\psi), D(\psi)$ depending only on $w = |\psi_A|^2 + |\psi_B|^2$

Nonlinear winding number

$$\nu = \frac{1}{2\pi i}\int dk\, \partial_k \log\left[\det\left(A(w) + e^{ik}D(w)\right)\right] \in \mathbb{Z}$$

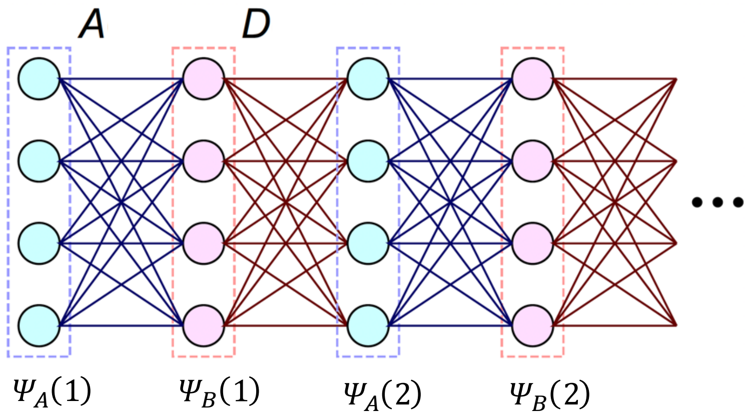
Proof of Bulk-Edge Correspondence in Linear Cases

$$E\Psi_A(x) = A^\dagger \Psi_B(x) + D^\dagger \Psi_B(x-1)$$

$$E\Psi_B(x) = A\Psi_A(x) + D\Psi_A(x+1)$$

Winding number

$$\nu = \frac{1}{2\pi i}\int dk\, \partial_k \log\left[\det\left(A + e^{ik} D\right)\right]$$



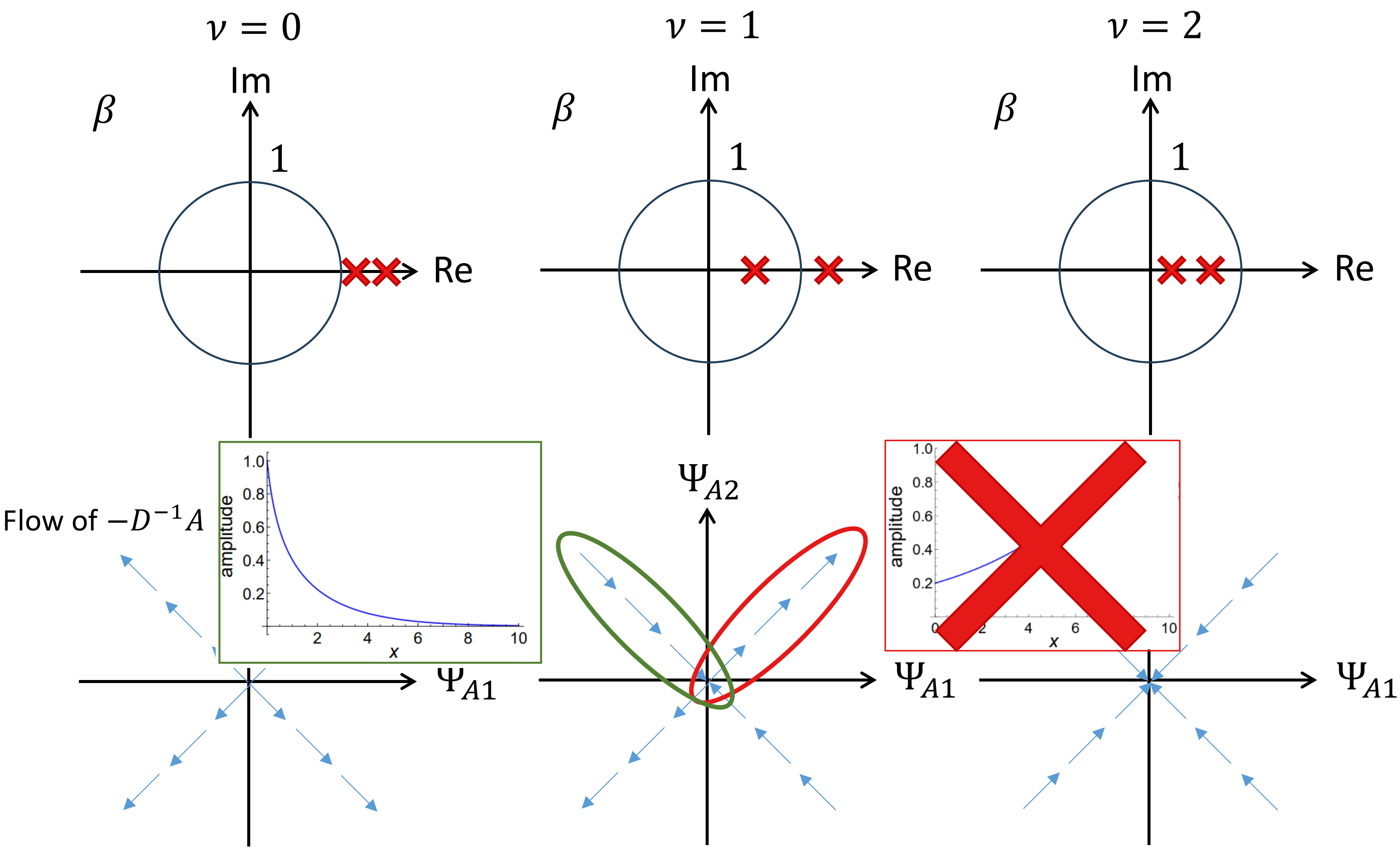
Fact 1 (argument principle)
Winding number
$\Leftrightarrow$ # of solutions β of $\det(A + \beta D) = 0$
satisfying $|\beta| < 1$

Fact 2
Solutions β of $\det(A + \beta D) = 0$
$\Leftrightarrow$ Eigenvalues of $-D^{-1}A$

Winding number
$\Leftrightarrow$ # of Eigenvalues of $-D^{-1}A$
satisfying $|\beta| < 1$

cf. Discussion on Green functions' poles
V. Gurarie PRB 83, 085426 (2011)

Proof of Bulk-Edge Correspondence in Linear Cases (2)

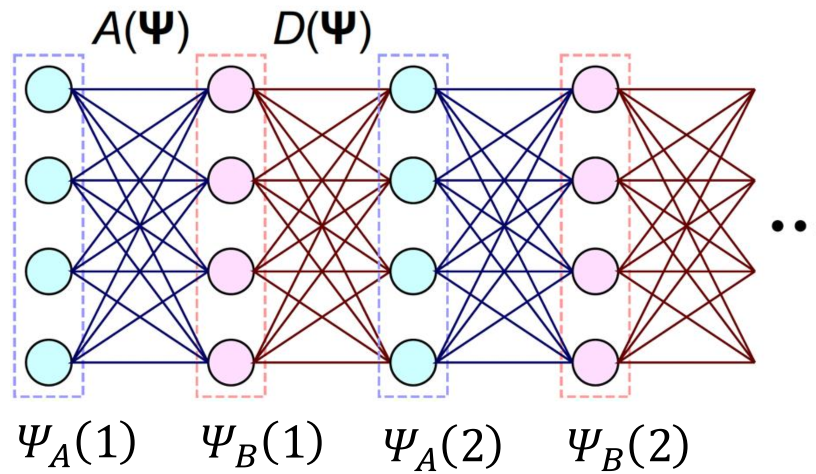


Corresponding to linearly independent edge modes

Extension to Nonlinear Cases

$$E\Psi_A(x) = A^\dagger(\Psi)\Psi_B(x) + D^\dagger(\Psi)\Psi_B(x-1)$$

$$E\Psi_B(x) = A(\Psi)\Psi_A(x) + D(\Psi)\Psi_A(x+1)$$

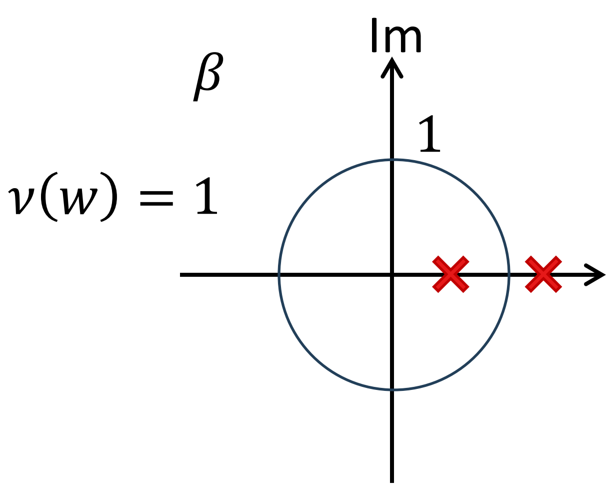


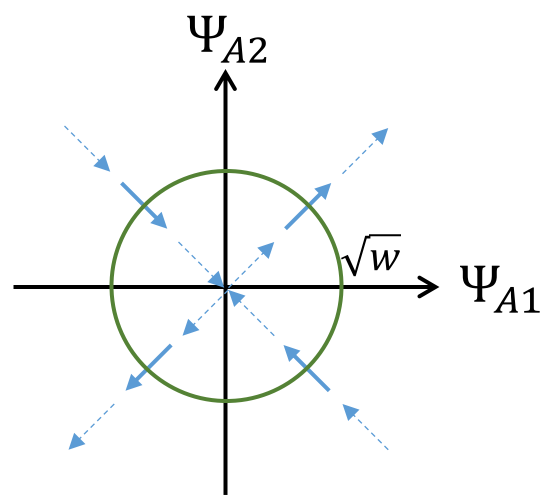
Nonlinear winding number

$$\nu = \frac{1}{2\pi i}\int dk\, \partial_k \log\left[\det\left(A(w) + e^{ik}D(w)\right)\right]$$

Winding number

$\Leftrightarrow$ # of Eigenvalues of $-D^{-1}(w)A(w)$ satisfying $|\beta| < 1$





Boundary condition

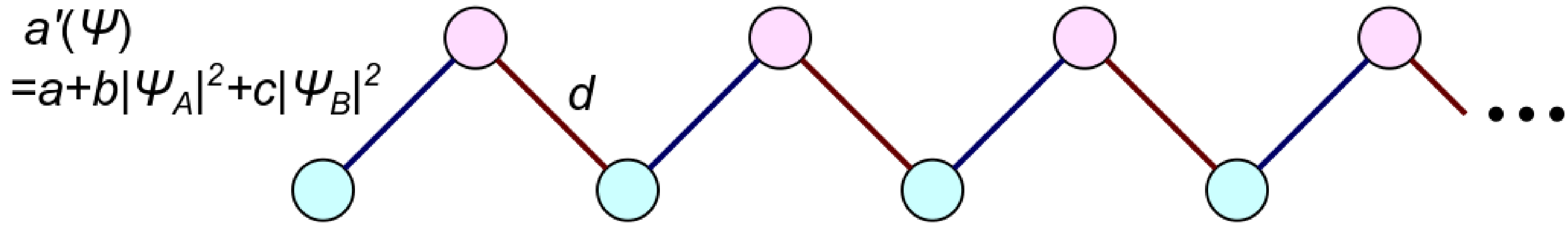
$\Psi_B(0) = 0, |\Psi_A(1)|^2 = w$

↓

Predict short-range amplification or attenuation

$|\Psi_A(1)| \lessgtr |\Psi_A(2)|$

Example: Nonlinear SSH Model (Nonlinear Hopping)



$$E\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \begin{pmatrix} 0 & a+b|\psi_A|^2+c|\psi_B|^2+de^{-ik} \\ a+b|\psi_A|^2+c|\psi_B|^2+de^{ik} & 0 \end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

Winding number

$$\nu = \frac{1}{2\pi i}\int dk\, \partial_k \log\left(a+bw+de^{ik}\right) \longrightarrow \nu = \begin{cases} 0 & \left(\left|a+bw_{\mathrm{edge}}\right| > |d|\right) \\ 1 & \left(\left|a+bw_{\mathrm{edge}}\right| < |d|\right) \end{cases}$$

Nonlinear transfer matrix

$$\Psi_A(x+1) = -\frac{a+b|\Psi_A(x)|^2}{d}\Psi_A(x), \qquad \Psi_B(x) = 0$$

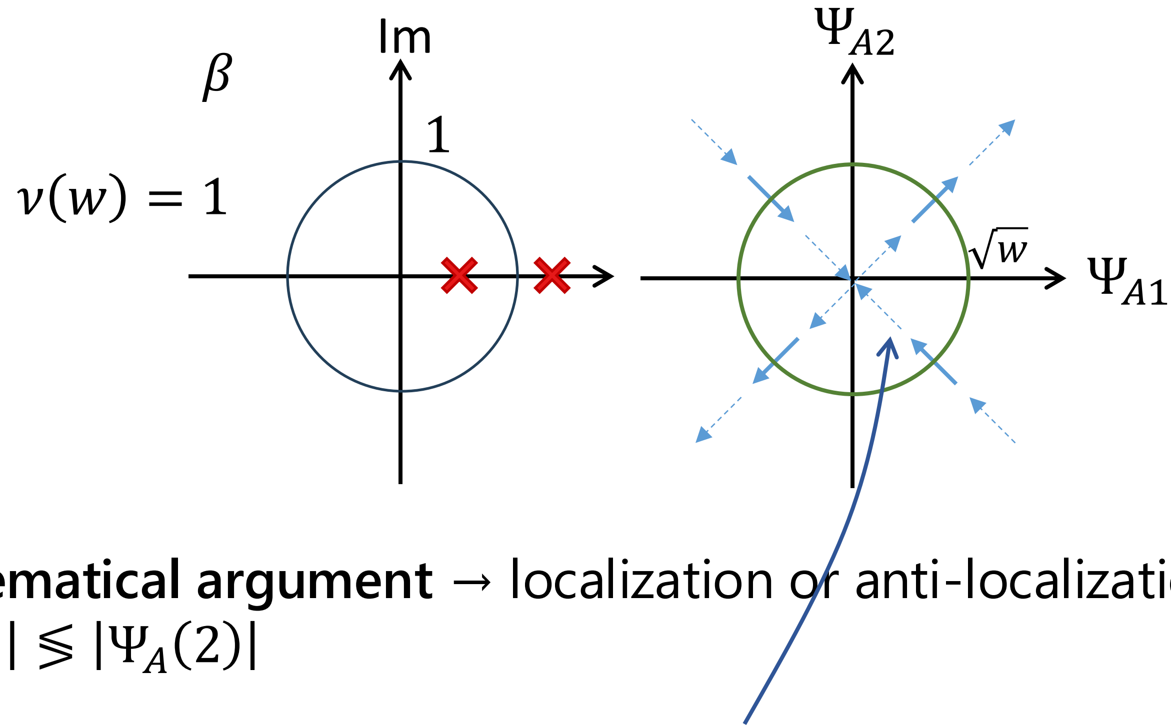
$$|\Psi_A(1)| > |\Psi_A(2)| \Leftrightarrow \left|a+bw_{\mathrm{edge}}\right| < |d| \Leftrightarrow \nu = 1$$

Corresponding to amplification or attenuation at the edge

Outline

- Introduction & Overview
- Definition of nonlinear bulk-edge correspondence
- Numerical demonstration
- Analysis from nonlinear transfer matrices
- **Bifurcation and nonlinear topological phenomena**
- Further extension

Nonlinear Effect: Long-Range Behavior

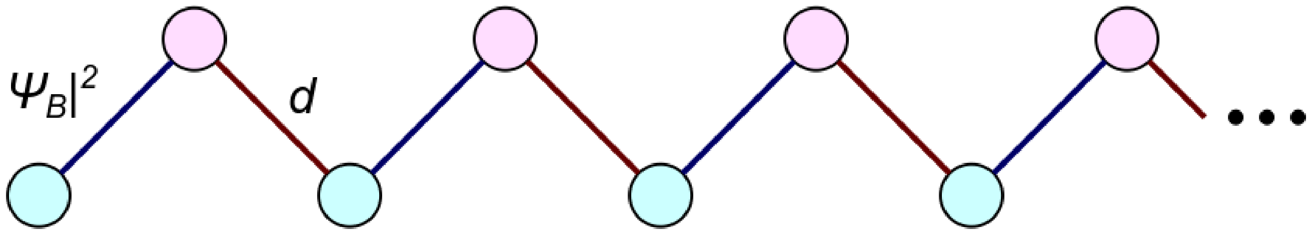


Mathematical argument → localization or anti-localization in a short range
$|\Psi_A(1)| \lesseqgtr |\Psi_A(2)|$

Q. Long-range behavior of zero modes $|\Psi_A(1)| \lesseqgtr |\Psi_A(\infty)|$
→ Fixed points, periodic orbits, or strange attractor induce
1. **nonlinearity-induced transitions,**
2. **breakdown of the bulk-edge correspondence.**

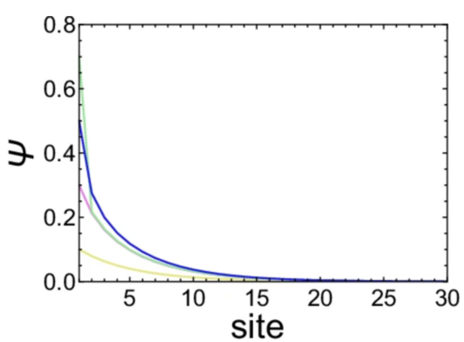
Chaos Transition in Nonlinear SSH Model

$a'(\Psi)$
$= a + b|\Psi_A|^2 + c|\Psi_B|^2$

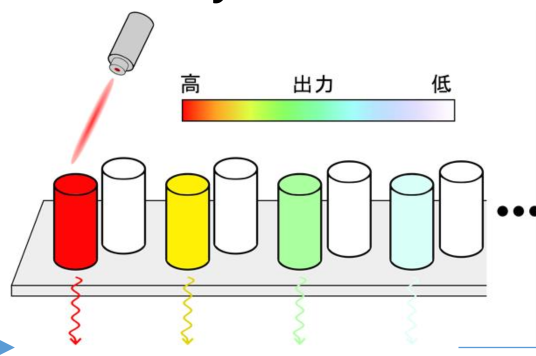


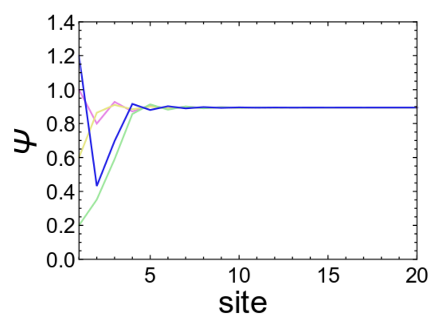
$$E\Psi_A(x) = (a + b|\psi_A(x)|^2 + c|\psi_B(x)|^2)\Psi_B(x) + d\Psi_B(x-1)$$
$$E\Psi_B(x) = (a + b|\psi_A(x)|^2 + c|\psi_B(x)|^2)\Psi_A(x) + d\Psi_A(x+1)$$

Linear limit

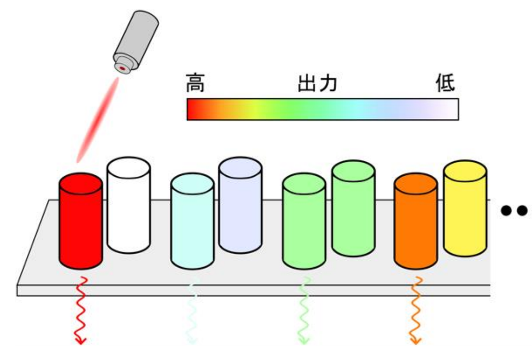


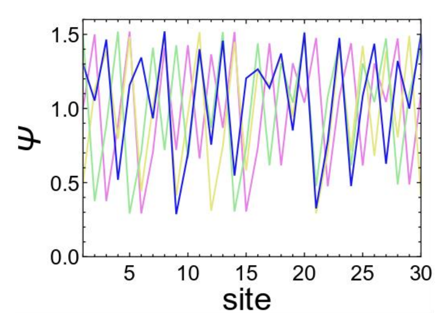
Nonlinearity-induced transition



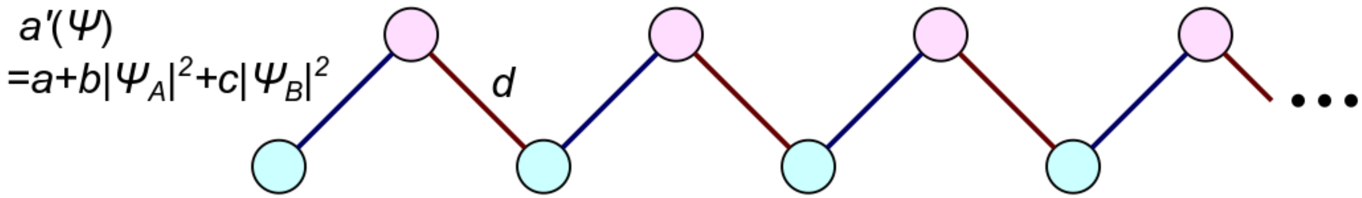


Spatial chaos





Bifurcation Diagram



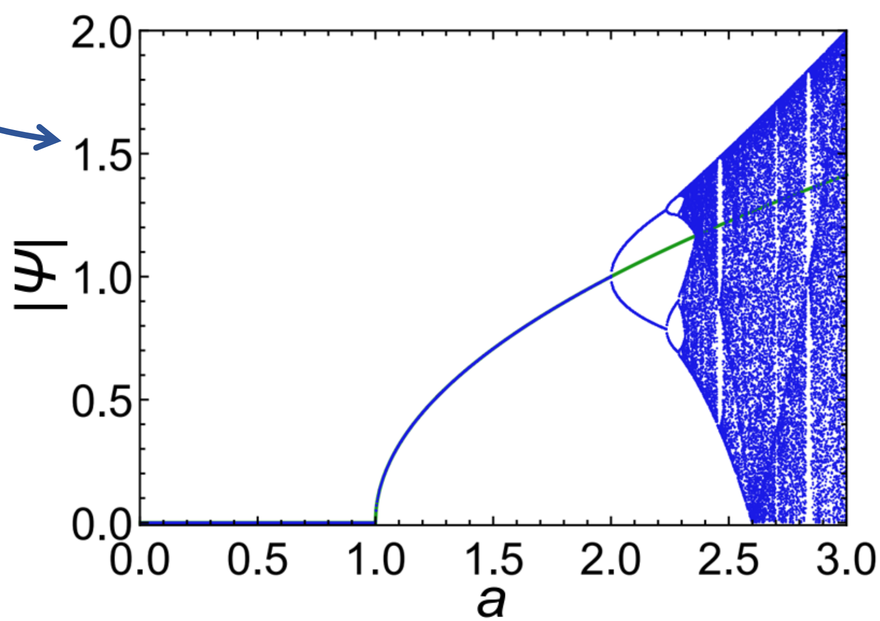
Spatial dynamics ($E = 0$, semi-infinite system ($x > 0$))

$$\Psi_A(x+1) = -\frac{a + b|\Psi_A(x)|^2}{d}\Psi_A(x), \qquad \Psi_B(x) = 0$$

Bifurcation diagram

Set of values of $\Psi_A(x)$ at $x \gg 1$

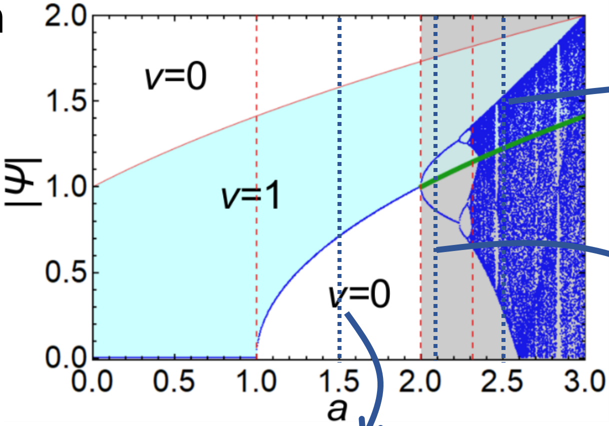
(= Fixed points, periodic orbits, and strange attractors)



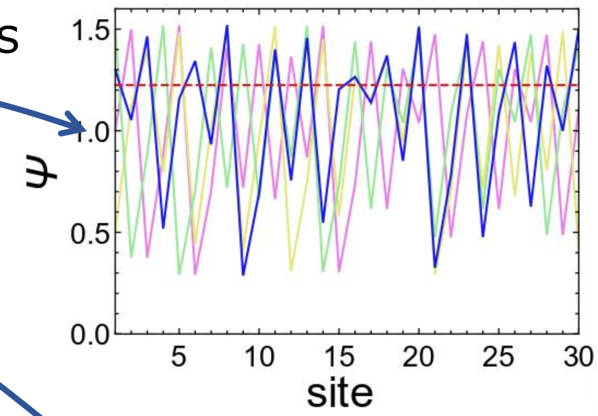
Bifurcation Diagram Corresponds to Nonlinear Phenomena

Bifurcation diagram

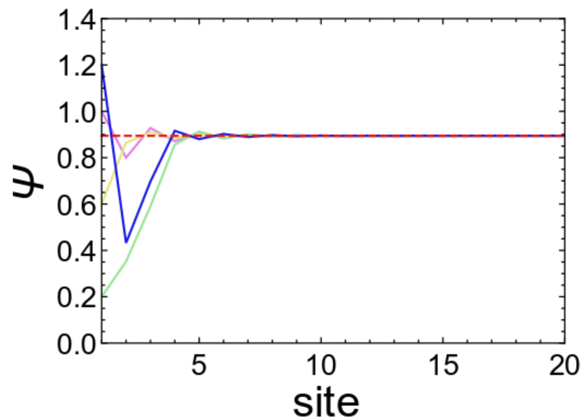
$$\Psi_A(x+1) = -\frac{a + b|\Psi_A(x)|^2}{d}\Psi_A(x)$$



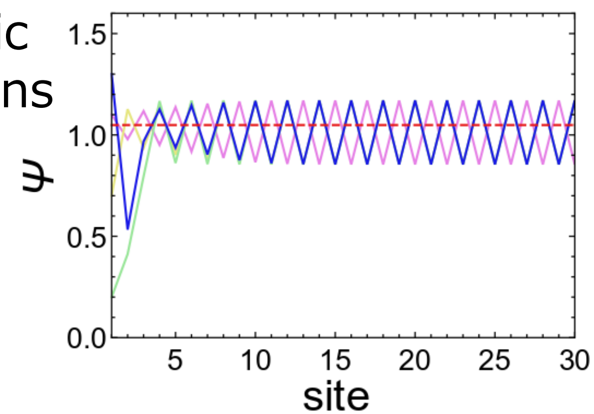
Chaos



Edge, anti-localized modes



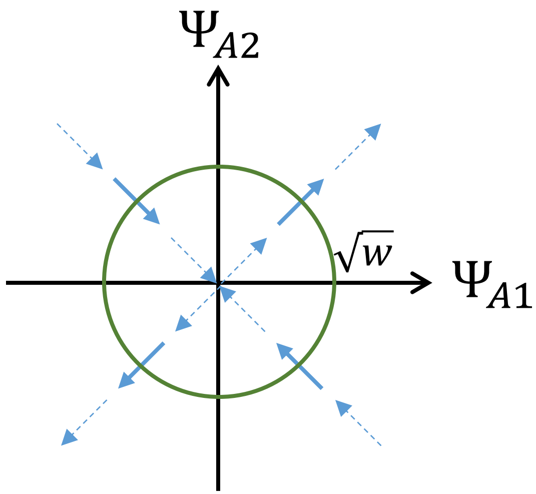
Periodic solutions



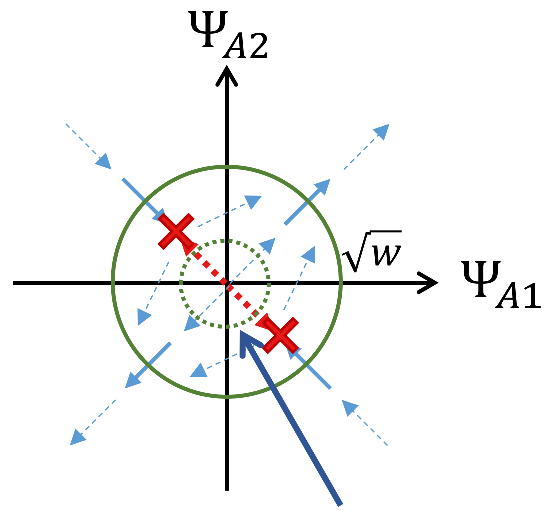
Unified picture of the nonlinear bulk-edge correspondence and its breakdown

First Step: Bifurcation and Nonlinearity-Induced Transition

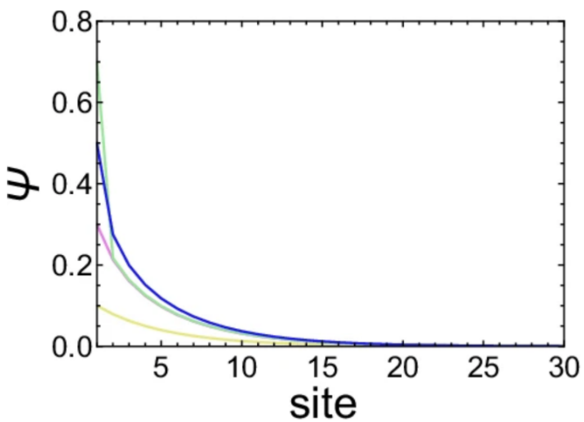
Linear limit



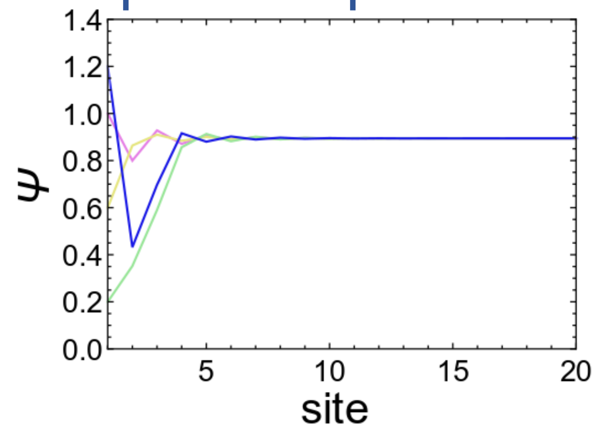
Nonlinearity-induced transition



Amplitude-dependent amplification



Converging to zero

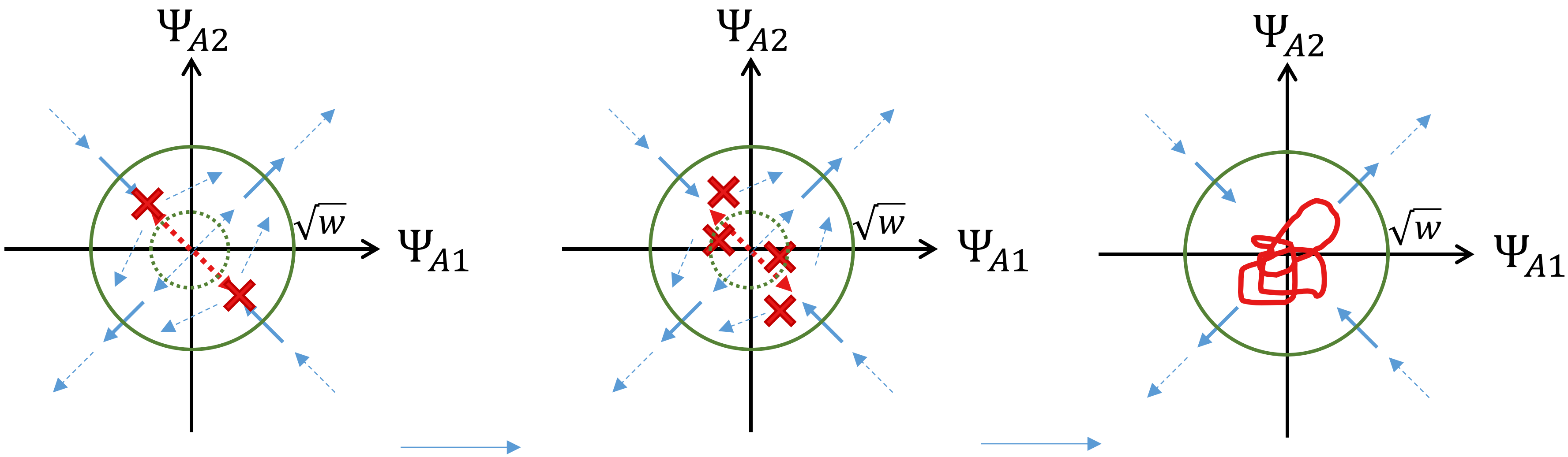


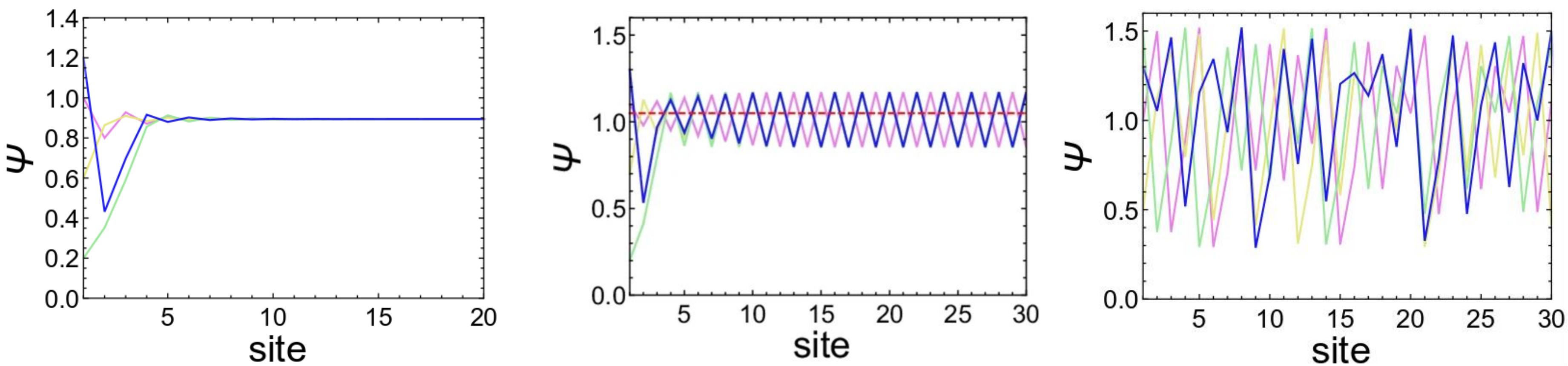
Converging to nonzero fixed points

Second Step: Chaos Transition

Nonlinearity-induced transition

Spatial chaos





Converging to nonzero fixed points

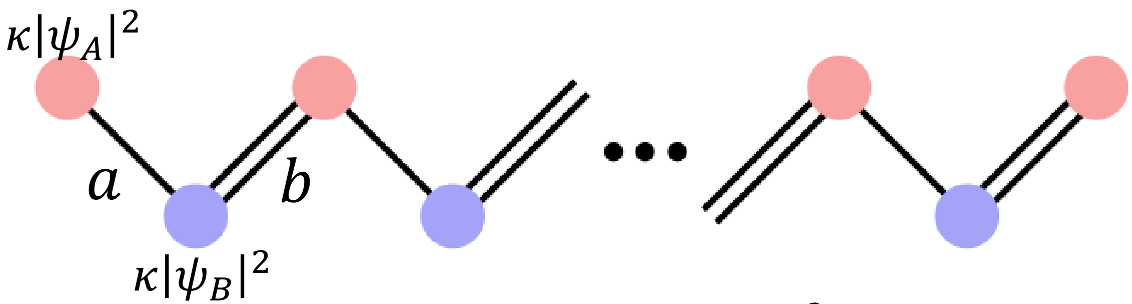
Converging to periodic solutions

No stable orbits

Outline

- Introduction & Overview
- Definition of nonlinear bulk-edge correspondence
- Numerical demonstration
- Analysis from nonlinear transfer matrices
- Bifurcation and nonlinear topological phenomena
- **Further extension**

Nonlinear SSH model with on-site nonlinearity



$a = -0.5, b = 1, \kappa = 1$

$$i\partial_t \Psi_A(x) = a\Psi_B(x) + b\Psi_B(x-1) + \kappa|\Psi_A(x)|^2$$
$$i\partial_t \Psi_B(x) = a\Psi_A(x) + b\Psi_A(x+1) + \kappa|\Psi_B(x)|^2$$

Wavenumber-space representation

$$E\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \begin{pmatrix}\kappa|\psi_A|^2 & a + be^{-ik}\\ a + be^{ik} & \kappa|\psi_B|^2\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

Time-reversal and space-inversion syms.
→ **Nonlinear Berry phase** is quantized.
cf. Zhou et al., Nat. Commun. 13, 3379 (2022)

Dynamical-system representation (generalization of a transfer matrix)

$$\Psi_A(x+1) = \frac{-a\Psi_A(x) - \kappa|\Psi_B(x)|^2\Psi_B(x) + \boxed{E}\Psi_B(x)}{b}$$

$$\Psi_B(x+1) = \frac{-b\Psi_B(x) - \kappa|\Psi_A(x+1)|^2\Psi_A(x+1) + \boxed{E}\Psi_A(x+1)}{a}$$

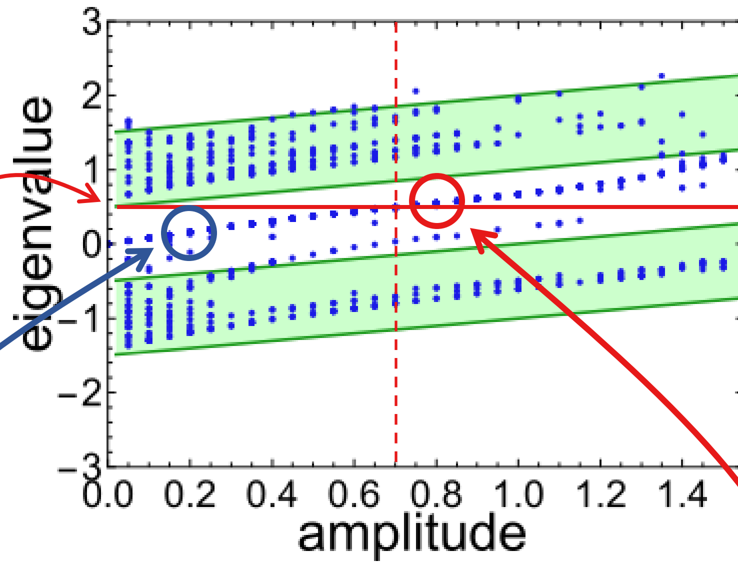
Remaining E dependence (broken sublattice symmetry)

Edge Modes: Transition to Non-vanishing Topological Modes

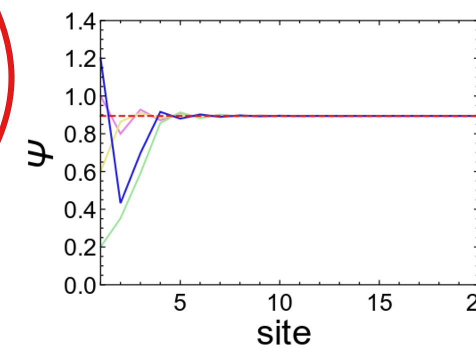
Transition at

$E_{\mathrm{edge}} = |a + b|$

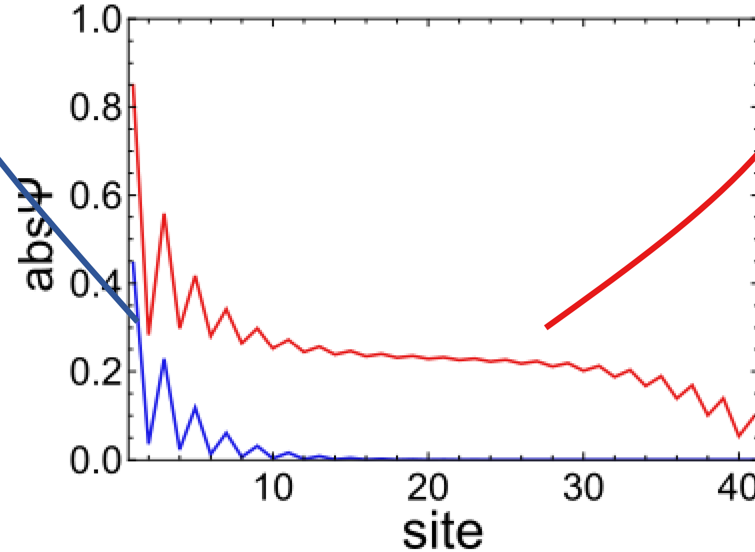
(No gap closing
→ same topology)



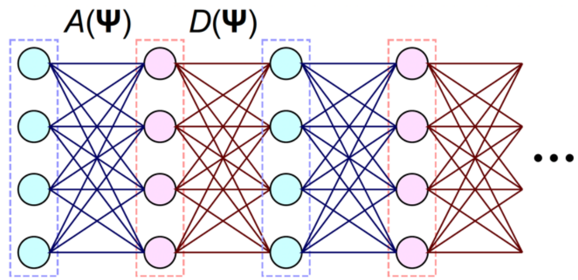
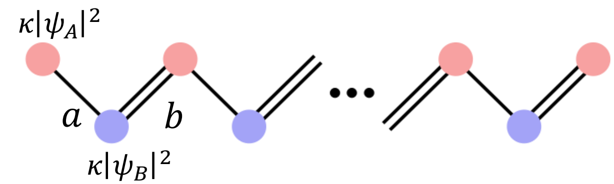
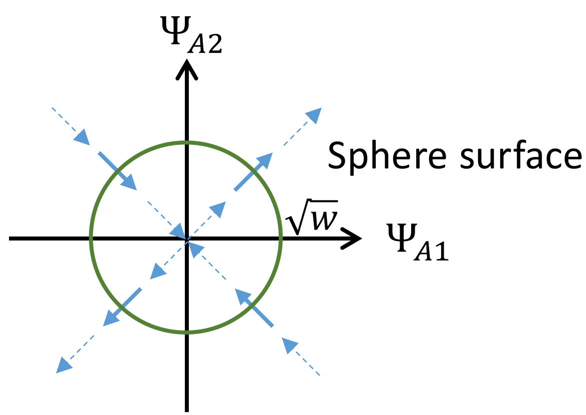
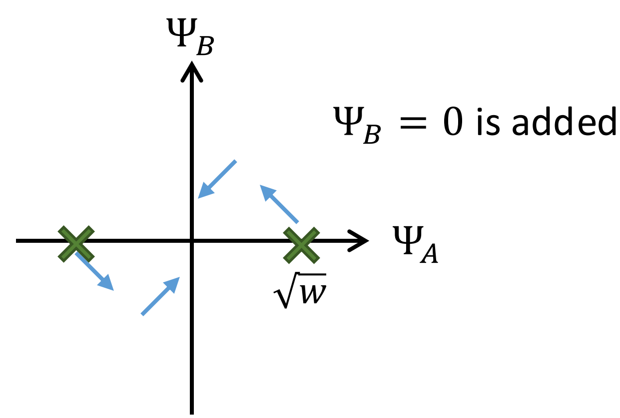
Similar to nonlinearity-induced topological edge mode (**non-vanishing mode**)



Similar to conventional topological edge modes (**decaying edge mode**)



Difference in Spatial Dynamics from Sublattice-Symmetric Cases

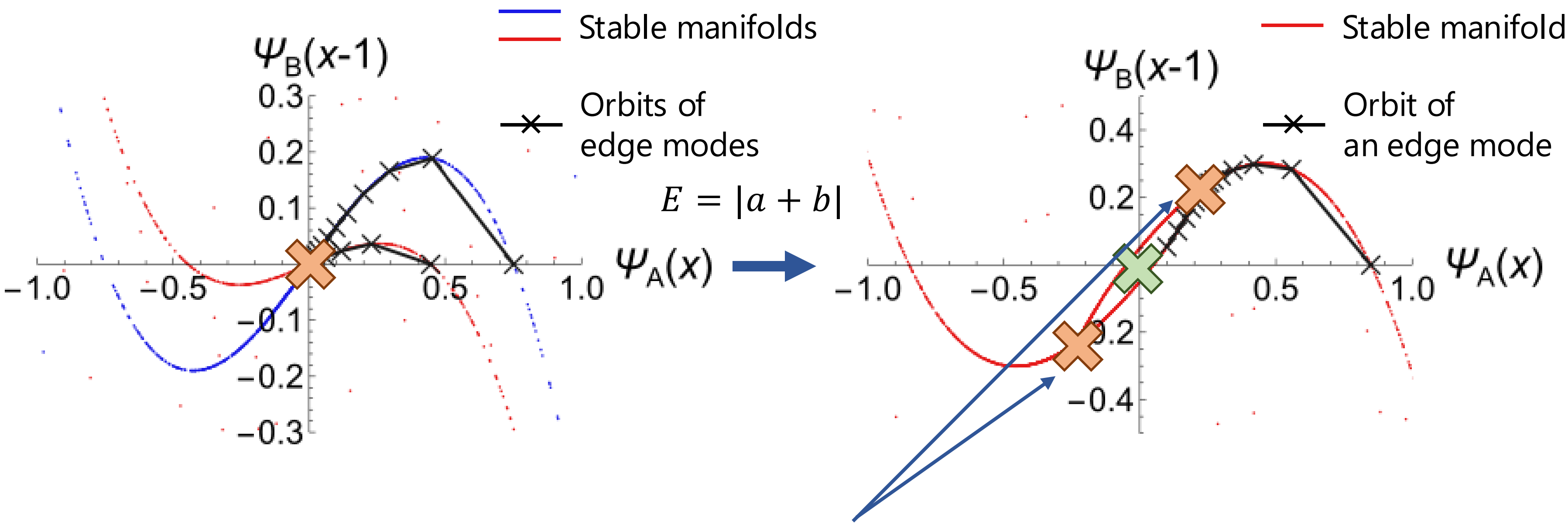
Nonlinear term	 Nonlinear coupling	 On-site
E-dependence	No	Yes (E is an additional parameter.)
Boundary condition		
Transition	Topological transition (with gap closing)	Without gap closing (Shift of E by nonlinearity)

Bifurcation in spatial dynamics by Changing E

Dynamical-system representation (cf. transfer matrix)

$$\Psi_A(x+1) = \frac{-a\Psi_A(x) - \kappa|\Psi_B(x)|^2\Psi_B(x) + E\Psi_B(x)}{b}$$

$$\Psi_B(x+1) = \frac{-b\Psi_B(x) - \kappa|\Psi_A(x+1)|^2\Psi_A(x+1) + E\Psi_A(x+1)}{a}$$

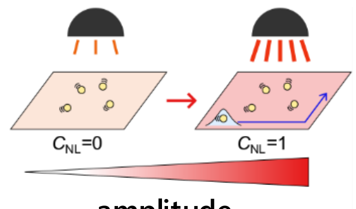


Emergence of stable fixed points (bifurcation)
→ transition to non-vanishing modes

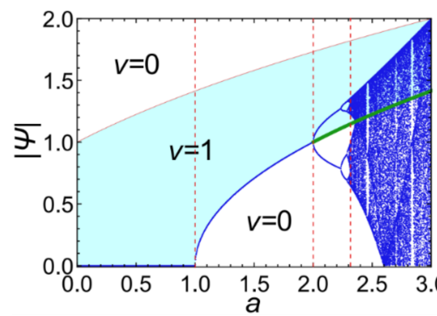
Summary

Nonlinear topology
→ Bulk-edge correspondence in nonlinear eigenvalue problem

Mathematically shown:
Topological invariant
↔ Short-range localization or anti-localization



Transfer matrix analysis:
Nonlinear effects in long-range behavior



Bifurcation→New fixed points, chaos attractors
- Nonlinearity-induced transition
- Breakdown of the bulk-edge correspondence

Prospect: experiments, symmetry, non-Hermitian+nonlinear

Refs. K. Sone, M. Ezawa, Y. Ashida, N. Yoshioka, and T. Sagawa, Nat. Phys. 20, 1164 (2024).
K. Sone, M. Ezawa, Z. Gong, T. Sawada, N. Yoshioka, and T. Sagawa, Nat. Commun. 16, 422 (2025).
K. Sone and Y. Hatsugai, arXiv:2501.10087.